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THE BEHAVIOUR OF THE MAJOR EXTRACTIVE INDUSTRIES IN LONG-RUN CLOSURES OF ORANI: A PROPOSAL

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ABSTRACT

In long-run closures of the ORANI computable general equilibrium model of Australia there is some tendency for the minerals industries to show a degree of volatility which is regarded as excessive by some commentators. If reliable guidelines about the values of partial equilibrium contemporaneous comparative static own-price elasticities of supply at a length of run of (say) 10 years were available, the introduction of industry-specific fixed factors ("the supplies of known reserves") into ORANI would correct the perceived problem. The present paper is mainly about exploring the limits that can be put on the relevant partial equilibrium elasticities by appealing to the theory of the optimal extraction of a resource under conditions where:

(a) known reserves are plentiful; but

(b) mines operate under conditions of cumulative cost increases as mining proceeds.

Numerical results at this stage are inconclusive. A research design to resolve the issue is put forward.

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The authors are grateful to Keith McLaren, who not only offered helpful general comments, but also clarified problems which arose when the first order conditions developed in Section 3 failed to be sufficient (despite appearances to the contrary). An outline of this contribution of his is reproduced in the Appendix.

1. BACKGROUND TO THE STUDY

The older class of computable general equilibrium (CGE) models, as exemplified in Evans (1972), relied heavily on linear programming. As a consequence, production possibility frontiers were linear, or piece-wise linear. In such circumstances, infinitesimal changes in relative prices could lead to implausibly large changes in product mixes. This is the so-called 'flip-flop', or overspecialization, problem.

The newer class of fully elaborated neo-classical CGE models, as exemplified in ORANI (Dixon, Parmenter, Sutton and Vincent (1982); hereafter DPSV), avoid this difficulty in short run closures. This is because there are many primary factors in the short run. These include industry-specific capital stocks. The production possibility frontier, moreover, is a smooth surface of appropriate convexity.

In long-run closures of ORANI, however, the number of primary factor constraints is many fewer. In particular, capital is assumed to be mobile between industries. Thus any long-run economy-wide capital constraint would not bind individual industries. In fact, it is only the land-using agricultural industries that have a fixed supply of an industry-specific factor in long-run closures. The potential problem is unlikely to surface for non-traded goods, since domestic demand conditions (and in particular, consumer demand where the values of inter-product substitution elasticities are typically less than one) will limit flip-flop. Similarly, for import-competing goods, Armington elasticities typically in the range 0.5 - 5.0 will tend to attenuate any tendency towards corner solutions. For non-land-using endogenous export industries, however, the potential problem of volatile output responses remains. The relevant commodities in ORANI are minerals or processed minerals (see Bruce (1985)):

Ferrous Metal Ores	(-10.0)
Non-Ferrous Metal Ores	(- 8.0)
Black Coal	(-20.0)
Basic Non-Ferrous Metals	(-10.0)

The number in parenthesis is the export demand elasticity in the parameter file of the 1977-78 (typicalized agriculture) ORANI database. The database does not recognize the existence of a factor specific to the industries producing these commodities which is fixed in the long run.

Having identified the potential problem, it should not be exaggerated. In Table 1 are reproduced results from Horridge (1985) demonstrating that the most volatile of all industries in long-run simulations of a tariff charge are miners. Nevertheless, it should be noted that the orders of magnitude involved are probably not so high as to be implausible a priori to all readers.

The aim of the present paper is to investigate the extent to which the theory of optimal extraction of a mineral may cast light on the above issue. Specifically, are there changes that should be made to the ORANI theory and/or database in order to make the long-run simulated behaviour of the mining industries more plausible? We attempt to answer this question in four stages. First, in Section 2, the relevant general analytical insights are reviewed. Then in Section 3, a special case, hopefully relevant to Australia, is developed in an operational form. Our first efforts to implement this operational form are reported in Section 4. Thence in Section 5 we reach the final stage of our analysis, wherein the implications of the role of mining in long-run closures of ORANI are explored. Finally, in Section 6, brief concluding remarks and a perspective for future research are offered.

$$\frac{\partial^2(PV)}{\partial x_2 \partial x_2} = F_x\Big]_{x_2} + F_y y'\Big]_{x_2} < 0 \ . \qquad (A.5)$$

Now from (3.6),

$$F_y\Big]_{x_2} = -\gamma^* e^{-rx_2} \ , \qquad (A.6)$$

while from (2.2), (3.2), (3.1) and (3.3),

$$F_x\Big]_{x_2} = \frac{\partial}{\partial x}\{e^{-rx}[Ke^{nx} y'(x) - \alpha - \theta y'(x) - \tfrac{1}{2}\beta[y'(x)]^2 - \gamma^* y(x)]\}\Big]_{x = x_2;\ y,\ y'\ \text{constant}}$$

$$= -rF\Big]_{x_2} + nKe^{(n-r)x_2} y'(x_2)$$

$$= nKe^{(n-r)x_2} y'(x_2) \quad (\text{since } F\Big]_{x_2} = 0 \text{ -- see Table 2}). \qquad (A.7)$$

Using (A.7) and (A.6) in (A.5), we obtain:

$$e^{-rx_2} y'(x_2)\,[nKe^{nx_2} - \gamma^*] < 0 \ . \qquad (A.8)$$

Since e^{-rx_2}, $y'(x_2) > 0$, to satisfy (A.5) we require:

$$\gamma^* > nKe^{nx_2} = np(x_2) \ ; \qquad (A.9)$$

that is,

$$n < \gamma^*/p(x_2) \ . \qquad (A.10)$$

If (A.10) is not satisfied, then we have located a minimum of PV with respect to x_2 (and thus a saddle point of the objective functional).

This failure seems to be an example of the situation in which the optimum to our free final time problem (with x_2 unknown) cannot be obtained by optimizing (with respect to the shut-down date, x_2) over a set of solutions with arbitrary known final times (see Seierstad (1984)). (We are indebted to Keith McLaren, both for the outline of the argument above, and for the reference.)

APPENDIX

Failure of the First-Order Conditions to Locate a Maximum Profit Plan in the Case of the Quadratic/Linear Cost Function -- Outline of a Proof by Keith R. McLaren

First, use the initial condition $y(0) = 0$ in (3.17b) to eliminate C_1 (in fact, $C_1 = -[C_2 + K/(n\beta)]$). Then y is a function of known parameters and of C_2 and x; write this

(A.1) $y = y(x;\ C_2)$.

The problem to be solved is to maximize the present value (PV) of the mine:

(A.2) $$\max_{\{C_2,\ x_2\}} \int_0^{x_2} F(x,\ y(x;\ C_2),\ y'(x;\ C_2))\,dx = \text{Max (PV)} .$$

(In the context of the assumptions made in Section 3, this is fully equivalent to (2.16).)

A first-order condition for (A.2) is:

(A.3) $$\frac{\partial(PV)}{\partial x_2} = F(x_2,\ y(x_2;\ C_2),\ y'(x_2,\ C_2)) = 0 .$$

Differentiating again, we obtain:

(A.4) $$\frac{\partial^2(PV)}{\partial x_2 \partial x_2} = F_x\Big|_{x_2} + F_y y'\Big|_{x_2} + F_{y'} y''\Big|_{x_2} ,$$

where F_x, F_y and $F_{y'}$, respectively, are the derivatives of F with respect to its 1st, 2nd and 3rd arguments. A second-order condition for locating a maximum of PV is that (A.4) be negative.

By (3.9) (an implication of the Euler Equation), the last term on the right of (A.4) vanishes; thus, for a maximum, we require:

Table 1

INDUSTRIES MOST AFFECTED IN LONG-RUN[(a)] ORANI SIMULATIONS BY A 25 PER CENT ACROSS-THE-BOARD TARIFF CUT

Industries	Deviation of Industry Output from Control in Year Projected (per cent)
Five industries with largest contraction in output:	
Man-made fibres, yarns	-14.9
Footwear	-10.6
Cotton yarns + fabrics	-8.7
Motor vehicles + parts	-8.0
Leather products	-4.7
Five industries with largest expansion in output:	
Non-ferrous metal ores	3.4
Ferrous metal ores	6.2
Black coal	6.5
Services to mining	7.2
Oil, gas + brown coal	10.5

(a) The long run here is defined by λ, the elasticity of Australian equity in the year projected with respect to saving in that year, which is set to 0.5. A possible calendar interpretation is of the order of 20 years. See Horridge (1985), especially p. 14.

Source: Horridge (1985), Table R18.

2. REVIEW OF THE CALCULUS-OF-VARIATIONS TREATMENT OF EXHAUSTIBLE RESOURCES

2.1 Basic Ideas

This review aims to give a detailed explanation of the principal results reported in Levhari and Liviatan (1977). Following Hildebrand (1965), the classical calculus of variations problem may be formulated as: find a continuously differentiable function y(x) which maximizes the functional

$$(2.1) \qquad I = \int_{x_1}^{x_2} F(x,y,y')\,dx \quad ,$$

subject to some end-point constraints (yet to be defined) involving x_1 and x_2. To fix ideas in the mining context, let x be (planning) time, y be cumulative extractions from a given deposit, and y' (≡ dy/dx) be the rate of extraction. Then if $x_1 = 0$ and

$$(2.2) \qquad F \equiv e^{-r(x-x_1)}\,\Pi(x,y,y') \quad ,$$

where r is the discount rate, and Π is the (instantaneous) rate of flow of profits, we can identify I as the present value of the profit stream.

The end-point constraints binding (2.1) may take a variety of forms, including:

(2.3) $y(x_1) = y_1$, $y(x_2) = y_2$; with the values of x_1, x_2, y_1, y_2 known;

and

(2.4) $y(x_1) = y_1$; $y(x_2) = g(x_2)$; with the values of x_1, y_1, and the form and parameters of g known.

In his classic article on the subject, Hotelling (1931) assumed (a) that the rate of output y' at the terminal point (y_2, x_2) is zero; and (b) that the deposit is exhausted at the terminal date; i.e., that $y_2 \equiv y(x_2) = a$, where a is the initial mineral endowment of the given deposit.

right of (5.3) consistent with a value of η(10) generated by the procedures discussed above in Sub-sections 5.1 and 5.2. Changes in the fortunes of the particular mining industry involved then show up as changes in the rental to the new fixed factor. (In a more ambitious framework, investment in exploration in the projected year would respond to such changes in long-run profitability.)

6. CONCLUDING REMARKS

Long-run supply elasticities in ORANI, as presently configured, may be implausibly high in the case of some mineral products. This paper has developed a model for the optimal exploitation of a mineral resource under conditions where:

(a) known reserves are plentiful; but

(b) mines operate under conditions of cumulative cost increases as mining proceeds.

At this stage, it is not yet known whether the meagre supply of information available will be enough to enable this model to put useful limits on contemporaneous comparative static supply elasticities at an arbitrary future date (say 10 years). A procedure for investigating this issue has been suggested above in Sub-sections 5.1 and 5.2. If the suggested procedure turns out to be feasible, then the database of long-run ORANI can easily be modified to make ORANI behave 'as if' the mining firms behave optimally along the lines described in Sections 2 and 3 of the paper. This amounts to introducing a new factor into the model which, like the different categories of agricultural land, is fixed, even in the long run -- this factor might be loosely thought of as "the known supply of reserves". By ascribing an appropriate share of value added to this factor in the model's database for the year projected, it is possible to ensure that the long-run comparative static behaviour of the mining industries is consistent with the micro theory developed in this paper.

Clearly, the proposals made in this paper are feasible, but a large volume of additional work is required.

(5.1) $v = (\alpha, \beta, \gamma^*, \theta)'$,

which would generate the given x_2. Sensitivity in cost parameter space might then be explored by using elasticities like those shown in Table 6. Thus with fixed K, r and n,

(5.2) $d \ln x_2 = \varepsilon_\alpha \, d \ln \alpha + \varepsilon_\beta \, d \ln \beta + \varepsilon_{\gamma*} \, d \ln \gamma^* + \varepsilon_\theta \, d \ln \theta$,

where the εs are the relevant elasticities. The v-subspace consistent with the given x_2 would then be spanned by taking differentials in α, β, γ^* θ consistent with a zero value for (5.2).

The sensitivity of $\eta(t)$ (with t = 10, say), over this subset of the parameter space would then be determined. (Hopefully the extrema would not be as wild as those found in Table 7.)

5.2 Aggregating Over Mines

If reasonably tight results are obtaiable from the strategy outlined above, the question still remains as to how aggregation to commodity level should take place. Ideally, this depends on the distribution of parameters across firms. This will not be estimable with any accuracy. Again, sensitivity analysis will be required.

5.3 Imbedding Partial Equilibrium Supply Elasticities into ORANI

Following Higgs (forthcoming, equation (7.6)), in ORANI the long-run own-price partial equilibrium supply elasticity for a commodity produced by a single industry may be approximated by:

(5.3) $\eta(10) \cong \sigma(1-S_F)/(S_F H_V)$,

where σ is the elasticity of substitution between primary factors (σ = 1.28 in the long-run ORANI database), S_F is the share of fixed factors in value added, and H_V is the share of value added in total costs. Note that as the share of fixed factors increases, the elasticity (5.3) declines. By introducing a new notional primary factor (the stock of known deposits), whose supply (in the model) is taken to be fixed in the long-run, we can vary S_F in the database to make the

He also assumed $x_1 = 0$ (as we will throughout, since this is but a convenient dating of the planning origin), and that $y(0) = 0$ (again harmless). Notice that Hotelling's end-point constraints correspond to neither of (2.3) nor (2.4), but to:

(2.5) $y(x_1) = y_1$; $y(x_2) = y_2$; $y'(x_2) = y_2'$ with known values of x_1, (i.e., zero), y_1 (also zero), y_2 (=a) and y_2' (zero again); but with x_2 unknown.

2.2 Lagrangean Functional

A general method of solution of the problem of determining y(x) is to replace I in (2.1) with a Lagrangean incorporating the resource endowment constraint,

(2.6) $$\int_0^{x_2} y'(x)\,dx \equiv y(x_2) \leq a.$$

The Lagrangean expression replacing I (see Hildebrand (1965, p. 142)) becomes:

(2.7) $$J = \int_0^{x_2} F(x,y,y')dx + \lambda \int_0^{x_2} \left[\frac{a}{x_2} - y'(x)\right]dx ,$$

where λ is a time-invariant, scalar multiplier. Let $H \equiv F - \lambda y'$; then

(2.7') $$J \equiv \int_0^{x_2} H\,dx + \lambda a \equiv \int_0^{x_2} F\,dx + \lambda(a - y(x_2)) .$$

The necessary conditions for maximising I subject to (2.6) are that the differential in J due to a variation in y should vanish, and that cumulative extractions do not exceed initial reserves; viz. that

(2.8) $$\delta J = \int_0^{x_2} \left[\frac{\partial H}{\partial y}(\delta y) + \frac{\partial H}{\partial y'}(\delta y')\right] dx = 0$$

and that

(2.9) $y(x_2) \leq a$,

where δy is the variation in the function y, and $(\delta y')$ is the induced variation in the derivative of that function; viz;

(2.10) $(\delta y') = \frac{d}{dx} (\delta y)$,

since the variation of a derivative is equal to the derivative of the variation (Hildebrand (1965, p. 133)). In (2.8), H is viewed as a function with explicit arguments x, y, y', and λ; i.e., $H \equiv H(x,y,y';\lambda)$ (see (2.7')). The partials in (2.8) are all evaluated at given values of x and λ; $\partial H/\partial y$ is evaluated with y' held fixed and $\partial H/\partial y'$ with y held fixed. That is, $\partial H/\partial y$ and $\partial H/\partial y'$ are to be viewed as the partial derivatives of H with respect to its second and third arguments, respectively.

Notice that

(2.11) $\frac{\partial H}{\partial y} \equiv \frac{\partial F}{\partial y}$,

while

(2.12) $\frac{\partial H}{\partial y'} = \frac{\partial F}{\partial y'} - \lambda$.

Substitution from (2.12) and (2.10) into (2.8) yields:

(2.13) $\int_0^{x_2} [\frac{\partial F}{\partial y} (\delta y) + \frac{\partial F}{\partial y'} \frac{d}{dx} (\delta y) - \lambda \frac{d}{dx} (\delta y)]dx = 0$.

Now

(2.14) $\int_0^{x_2} \lambda \frac{d}{dx} (\delta y)dx = \lambda [\delta y]_0^{x_2}$.

It is part of the variational method that δy should vanish identically at any known end points; thus $\delta y(0) = 0$. Hence (2.13) becomes:

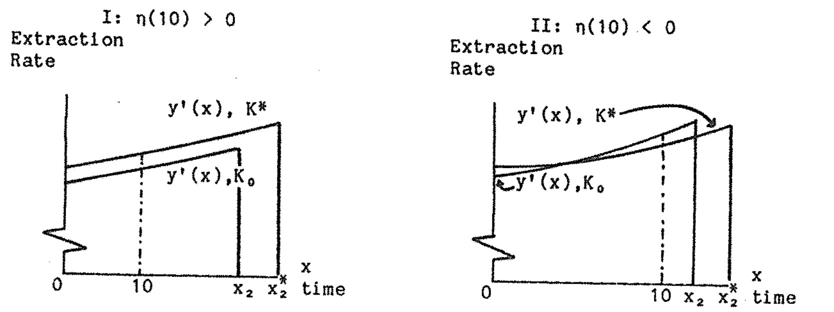

Figure 3 The Possible Effects of an Expected Sustained Increase in the Mineral Price on the Optimal Extraction Path
The initial expected price path is $p(x) = K_0 e^{nx}$; the shocked path is $p(x) = K^* e^{nx}$, where $K^* = 1.01 K_0$.

5. PROPOSED APPLICATION TO ORANI

5.1 Narrowing Down the Range of Relevant Possibilities

The foregoing results are disappointing to the extent that the contemporaneous comparative static price elasticities of supply from a mine at an arbitrary future date cannot even be signed without specific information on parameters. Since the mining sector in Australia is relatively highly concentrated, for reasons of commercial secrecy it is unlikely that engineering and/or cost information which would allow the estimation of mine-specific cost functions will be obtainable.

One approach for further work is as follows. Since the cost parameters α, γ^*, θ and β are not likely to be estimable, we could concentrate instead on the planned shut-down date x_2. This is a quantity about which general information may be obtainable. As we have seen above, treating $K = p(0)$ as arbitrary results in no loss of generality (and in any event, the current price of the mineral is known). The market discount rate r, and the expected rate of price increase for the mineral, n, should be more of less discernible from market data and trade literature. Treating K, r, n and x_2 as known, by a process of trial and error we could find an arbitrary cost parameter vector

Table 8

CHARACTERISTICS OF A PARAMETER SET POSSESSING A NEGATIVE ELASTICITY FOR THE RATE OF EXTRACTION IN TEN YEARS TIME WITH RESPECT TO THE SUSTAINED PRICE LEVEL OF THE MINERAL

Parameter	Value	Elasticity of x_2 with Respect to Parameter Shown in Column 1
K	104.50	4.9672
α	200,000.00	-1.4835
γ^*	1.96	-1.0539
Θ	20.0	-0.9293
r	0.01	0.0042
n	0.001	0.0743
β	0.01	-1.4815
Solution x_2	11.5 years	

Time x	Elasticity of y'(x) with Respect to K
0	0.0751
0.6	0.0654
1.2	0.0559
1.8	0.0467
2.4	0.0378
3.0	0.0290
3.6	0.0206
4.2	0.0123
4.8	0.0043
5.4	-0.0036
6.0	-0.0123
6.7	-0.0187
7.3	-0.0260
7.9	-0.0331
8.4	-0.0400
9.1	-0.0467
9.7	-0.0534
10.0	-0.0568
10.1	-0.0598
10.9	-0.0661
11.5 (x_2)	-0.0723

(2.15) $$\int_0^{x_2}\left[\frac{\partial F}{\partial y}(\delta y) + \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)\right] dx \;-\; \lambda\, \delta y(x_2) \;=\; 0.$$

δy does not have to vanish at the (unknown) planning horizon length x_2.

Formally, we seek the solution to the following problem:

(2.16) Subject to $y(0) = 0$, find a function $y(x)$ and a length of planning horizon x_2 which jointly maximize J in (2.7), while simultaneously finding the value of a Lagrange multiplier λ, which ensures that $y(x_2) \leq a$.

2.3 First Order Conditions

J should be viewed as a functional in y and as a function of λ and x_2. The necessary conditions are found by putting the partial derivative of J with respect to x_2 to zero and with respect to λ to a value not less than zero; and by putting to zero the variation in J with respect to y. That is, we require y, x_2 and λ such that

(2.17a) $$\frac{\partial J}{\partial \lambda} \geq 0 \;,$$

(2.17b) $$\frac{\partial J}{\partial x_2} = 0 \;,$$

and

(2.17c) $$\delta J \equiv \frac{\partial J}{\partial y}(\delta y) = 0 \;.$$

The middle term of (2.17c) may be decomposed as follows (yielding the left-hand side of (2.15) again):

(2.18) $$\frac{\partial J}{\partial y}(\delta y) = \int_0^{x_2}\left[\frac{\partial F}{\partial y}(\delta y) + \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)\right]dx - \lambda\, \delta y(x_2).$$

Let $H^\circ(x,\lambda) \equiv H(x, y(x), y'(x), \lambda)$. Since

$$J = \int_0^{x_2} H^\circ(x,\lambda)dx + \lambda a = \left[h^\circ(x,\lambda)\right]_0^{x_2} + \lambda a$$

(2.19) $$= h^\circ(x_2,\lambda) - h^\circ(0,\lambda) + \lambda a \;,$$

where $\partial h^{o}/\partial x \equiv H^{o}$, it follows from (2.17b) that

$$\frac{\partial J}{\partial x_2} = \frac{\partial}{\partial x_2} h^{o}(x_2, \lambda)$$

(2.20) $$= H^{o}(x_2, \lambda) = H\Big]_{x=x_2} = 0 \ .$$

Since F is not a function of λ, from (2.7') we see that

(2.21) $$\partial J/\partial\lambda = [a - y(x_2)] \ ;$$

thus (2.17a) yields (2.6) again.

The expression $[\partial F/\partial y'] \frac{d}{dx} (\delta y)$ contained on the right of (2.18) may be integrated by parts to yield:

(2.22) $$\int \frac{\partial F}{\partial y'} \frac{d}{dx} (\delta y)\, dx = \frac{\partial F}{\partial y'} (\delta y) - \int (\delta y) \frac{d}{dx} \left(\frac{\partial F}{\partial y'}\right) dx \ .$$

Since $\delta y(0) = 0$,

(2.23) $$\left[\frac{\partial F}{\partial y'} (\delta y)\right]_0^{x_2} = \frac{\partial F}{\partial y'}\Big]_{x=x_2} \delta y(x_2) \ .$$

Using (2.22) and (2.23) in substitutions into (2.17b), we obtain:

(2.24) $$\delta J = \int_0^{x_2} \left\{\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'}\right)\right\}(\delta y)dx + \left[\left(\frac{\partial F}{\partial y'} - \lambda\right)\delta y\right]_{x=x_2} = 0 \ .$$

Kuhn-Tucker tells us that exactly one of the following holds:

(2.25a) $a > y(x_2),\ \lambda = 0$;

or

(2.25b) $\lambda > 0,\ a = y(x_2)$.

If (2.25a) holds, (2.24) reduces to

Table 7

RESPONSE OF SUPPLY IN THE LONG RUN (10 YEARS) TO A SUSTAINED CHANGE IN THE EXPECTED PRICE OF THE MINERAL[(a)]

Search over Parameter Set A

Number of possible combinations tried: 729
Number of valid solutions found for x_2: 367[(b)]

Elasticity of y'(10) with Respect to K	
Minimum Elasticity	-0.0568[(c)]
Maximum Elasticity	1.4636[(d)]
Average Elasticity	0.9308
Mean Absolute Deviation	0.2616

Search over Parameter Set B

Number of possible combinations tried: 729
Number of valid solutions found for x_2: 729

Elasticity of y'(10) with Respect to K	
Minimum Elasticity	0.0429[(e)]
Maximum Elasticity	1.5035[(f)]
Average Elasticity	0.9823
Mean Absolute Deviation	0.2416

(a) This table should be read in conjunction with the multiple parameter sets listed in Table 5.

(b) The remaining combinations gave values for x_2 of less than ten years, and have thus been excluded from analysis.

(c), (d), (e), (f): The parameter sets generating these extrema were:

	K	α	Y*	θ	r	n	β
(c)	104.5	200,000	1.96	20.0	0.01	0.001	0.01
(d)	104.5	50,000	0.010	20.0	0.01	-0.05	0.00001
(e)	150.0	200,000	5	20.0	0.01	0.001	0.01
(f)	150.0	50,000	1	20.0	0.01	0.001	0.00001

sets A and B, respectively. Experiments over wider parameter sets suggest that typically, the elasticity of $y'(10)$ with respect to K falls in the range of 0.5 to 3. This is not the case, however, for the minima reported in Table 7. In the case of Parameter Set A, a single negative elasticity was found. It must be that

$$\gamma^* \, e^{r(x-x_2)} \frac{\partial x_2}{\partial K} \quad > \quad e^{nx} \quad ,$$

(see equations (3.34) and (3.37)). The specific set which provides the unexpected negative elasticity is listed in Table 8 along with a series of elasticities of $y'(x)$ with respect to K throughout the plan. An increase in K, ceteris paribus, will not only increase the initial extraction rate, but will also push out the optimal shut-down date, x_2. Two possible alternatives are illustrated in Figure 3. Panel I represents the typical case whilst Panel II is representative of the case reported in Table 8. The salient feature of the parameter set corresponding to Panel II of Figure 3 is a shut-down date very soon after period 10. In such a case the elasticity of x_2 with respect to K will be relatively high. Indeed, in the instance of the parameter set shown in Table 8, it is 4.967, which is considerably larger than any of the relevant elasticities reported in Table 6.

(2.26) $$\delta J = \left[\frac{\partial F}{\partial y'} \right]_{x=x_2} \delta y(x_2) + \int_0^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right\} (\delta y) dx = 0 \ .$$

In such a case, reserves are not exhausted at the end of the planning horizon (when production ceases). An additional unit of reserves would not command a positive market price under these conditions. This does not gainsay the profitability of extractions of ore at the optimal rate in the foreground of the plan. However, since some reserves are left at the end of the plan, and since any new reserves (of the same grade, in the same location) would not be required during the life of the plan, their present value, like that of excess existing reserves, is zero. The necessary condition (2.26) is, as we would require, exactly the same as the necessary condition for an unconstrained problem in which there is no requirement such as (2.6) (see Hildebrand (1965), p. 145).

(2.24) must be true for an arbitrary variation δy. Since y will be a function characterized by a finite n-dimensional vector of parameters μ, we can think of δy being generated by a vector of differentials $d\mu$. Since the elements μ_i of μ are parameters, no non-trivial constraint of the form $\Sigma \, c_i \, d\mu_i = k$ (where the c_i and k are constants) applies. If $y(x)$ is written as the function $\psi(\mu;x)$ of its own parameters, then

(2.27) $$\delta y(x) = \sum_{i=1}^{n} \left[\frac{\partial}{\partial \mu_i} \psi(\mu;x) \right] d\mu_i \ .$$

Write

(2.28a) $$\left. \frac{\partial F}{\partial y'} \right|_{x=x_2} = A(x_2) \quad ,$$

and

(2.28b) $$\left. \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right\} \right|_{x} = B(x) \ .$$

Then (2.24) is

(2.29) $$[A(x_2) - \lambda] \sum_{i=1}^{n} \frac{\partial}{\partial \mu_i} \psi(\mu;x_2) \, d\mu_i$$

$$+ \sum_{i=1}^{n} \left[\int_0^{x_2} B(x) \frac{\partial}{\partial \mu_i} \psi(\mu;x)dx \right] d\mu_i = 0 \quad ;$$

that is, with an obvious change of notation,

$$(2.30) \qquad \sum_{i=1}^{n} (\tilde{A}_i + \tilde{B}_i) d\mu_i = 0 \quad .$$

But, since the μ_i are free parameters, the only constraint of the kind (2.30) which is permissible is the trivial one in which $\tilde{A}_i + \tilde{B}_i = 0$.

That is ,

$$(2.31) \qquad [A(x_2) - \lambda] \frac{\partial}{\partial \mu_i} \psi(\mu,x_2) = - \int_0^{x_2} B(x) \frac{\partial}{\partial \mu_i} \psi(\mu,x)dx \ (i=1,\ldots,n).$$

Differentiate (2.31) with respect to x_2, obtaining:

$$(2.33) \qquad [A'(x_2)+B(x_2) - \lambda'_{x_2}] \frac{\partial}{\partial \mu_i} \psi(\mu,x_2)$$

$$= [\lambda - A(x_2)] \frac{\partial^2}{\partial \mu_i \partial x_2} \psi(\mu,x_2) \ (i=1,..,n) \ ,$$

where λ'_{x_2} is the derivate of λ with respect to x_2. But

$$B(x_2) = \left.\frac{\partial F}{\partial y}\right|_{x_2} - \left.\frac{d}{dx} A(x)\right|_{x_2}$$

$$(2.34) \qquad = \left.\frac{\partial F}{\partial y}\right|_{x_2} - A'(x_2) \quad .$$

Substituting from (2.34) into (2.33):

$$(2.35) \qquad \left[\left.\frac{\partial F}{\partial y}\right|_{x_2} - \lambda'_{x_2}\right] \frac{\partial}{\partial \mu_i} \psi(\mu,x_2)$$

$$= \left[\lambda - \left.\frac{\partial F}{\partial y'}\right|_{x_2}\right] \frac{\partial^2}{\partial \mu_i \partial x_2} \psi(\mu,x_2) \quad (i=1,\ldots,n) \quad .$$

Suppose

$$\left.\frac{\partial F}{\partial y}\right|_{x_2} \neq \lambda'_{x_2} \ , \quad \left.\frac{\partial F}{\partial y'}\right|_{x_2} \neq \lambda \ .$$

Then

In Table 6 the elasticities of the shut-down date with respect to the parameters of the problem are reported for a subset of eight parameter combinations. Because of its role in computing $\eta(t)$ (see (3.34)), the first column of Table 6 is of particular interest.

Table 6

ELASTICITIES OF THE SHUT-DOWN DATE WITH RESPECT TO PARAMETERS OF THE PROBLEM FOR SPECIFIC NUMERICAL SOLUTIONS(a)

Parameter Set	Elasticity of x_2 with Respect to:						
	K	α	Y*	θ	r	n	β
Table 4							
Set I	0.5923	-0.0008	-0.4660	-0.1242	-0.0156	-0.5177	-0.0008
Set II	0.6954	-0.0004	-0.5831	-0.1103	-0.0356	-0.3800	-0.0006
Set III	2.0313	-0.4923	-0.9557	-0.2288	-0.1194	0.0055	-0.4936
Table 5							
Set AI	0.1911	-0.0163	-0.1579	-0.0005	-0.0002	-0.8410	-0.0163
Set AII	1.2538	-0.2645	-0.5579	-0.1662	-0.0209	-0.4207	-0.2645
Set AIII(b)	-	-	-	-	-	-	-
Set BI	0.9562	0.0000	-0.9542	-0.00003	-0.1491	0.1045	0.0003
Set BII	1.1660	0.0000	-0.8992	-0.0895	3.2487	0.0001	3.3522
Set BIII	1.2266	-0.0726	-0.9211	-0.1593	-0.1088	0.0321	-0.0723

(a) This table should be read in conjunction with Tables 4 and 5.

(b) Not computed.

4.4 Comparative Static Long-Run Price Elasticities of Supply

All possible parameter combinations selected from the multiple sets listed in Table 5 have been tested for valid solutions to x_2, and the elasticity $\eta(10)$ of the rate of extraction in year 10, $y'(10)$, with respect to K has been computed. Table 7 records an analysis of these elasticities for the 367 and 729 valid solutions found for parameter

Table 5

TWO MULTIPLE PARAMETER SETS USED TO COMPLETE A GRID SEARCH FOR VALID SHUT-DOWN DATES, x_2(a)

Parameter	Parameter Set A(b) I	II	III
K	104.50(c)	104.50(c)	104.50(c)
α	200,000.00	100,000.00	50,000.00
γ^*	0.01	1.96	5.0
Θ	0.01	10.0	20.0
r	0.01	0.07	0.1
n	-0.05	-0.02	0.001
β	0.00001	0.00743	0.01
Solutions(d)			
x_2	55.4 years	12.6 years	<10 years

Parameter	Parameter Set B(b) I	II	III
K	150.00(c)	150.00(c)	150.00(c)
α	200,000.00	100,000.00	50,000.00
γ^*	1.0	1.96	5.0
Θ	0.01	10.0	20.0
r	0.01	0.07	0.1
n	0.0005	0.000001	0.001
β	0.00001	0.00743	0.01
Solutions(d)			
x_2	109.0 years	37.9 years	15.5 years

Notes: (a) The grid searches in each case are over the 729 (i.e., 3^6) combinations of values of parameters selected, respectively, from Parameter Set A and Parameter Set B.

(b) For units, see Table 4.

(c) The initial price is used as a numeraire and has thus not been altered between columns.

(d) The x_2 solutions shown (6 out of 2 x 729) correspond to the parameter vector immediately above the shown x_2 value.

(2.36) $$\frac{\partial}{\partial\mu_i}\psi(\mu,x_2)/\frac{\partial^2}{\partial\mu_i\partial x}\psi(\mu,x_2) = K \qquad (i=1,\ldots,n) ,$$

where K is independent of i. This is an unreasonably severe restriction on ψ and therefore on the permissible class of extremals y. This strongly suggests that:

(2.37) $$\left.\frac{\partial F}{\partial y}\right|_{x_2} = \lambda'_{x_2} , \qquad \left.\frac{\partial F}{\partial y'}\right|_{x_2} = \lambda .$$

Using the second part of (2.37) in (2.24), we obtain:

(2.38) $$\delta J = \int_0^{x_2} \{\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'})\}\delta y \, dx = 0 .$$

Since δy must be allowed to be arbitrary, (2.38) implies the <u>Euler Equation</u>

(2.39) $$\frac{\partial F}{\partial y} = \frac{d}{dx}(\frac{\partial F}{\partial y'}) \qquad \text{for } x \in [0,x_2] .$$

The other necessary conditions derived above are repeated here for convenience:

(2.20) $$H\rfloor_{x_2} = 0 = F(x_2) - \lambda\, y'(x_2) ;$$

(2.37) $$\lambda = \left.\frac{\partial F}{\partial y'}\right|_{x_2} ;$$

(2.25a) $$a > y(x_2) , \quad \lambda = 0 ;$$

(2.25b) $$a = y(x_2) , \quad \lambda > 0 .$$

Substituting (2.37) into (2.20), we obtain

(2.40) $$F(x_2) = \left.\frac{\partial F}{\partial y'}\right|_{x_2} y'(x_2) .$$

The present value (PV) of the profit flow at the terminal point is equal to the marginal effect on the present value of profit due to a small change in the extraction rate at that point, multiplied by the extraction rate. (2.40) implies that

(2.41a) Average PV Profit at end-point = Marginal PV Profit at end-point ;

i.e., when $y'(x_2) \neq 0$,

$$F(x_2)/y'(x_2) = \left.\frac{\partial F}{\partial y'}\right]_{x_2} ; \quad (2.41b)$$

thus in the case of surplus reserves, average PV profit is maximized at the terminal point. When $y'(x_2)$ is zero, (2.41b) continues to hold in a limiting sense as $y'(x_2) \to 0$ (see Levhari and Liviatan (1977, pp. 180 and 181). If there is no shortage of reserves, and (2.25a) applies, the marginal PV profit at the terminal point, $[\partial F/\partial y']_{x=x_2}$, is driven to zero, so that $F(x_2) = 0$. These results, together with results for the case in which x_2 is known, are summarized in Table 2.

2.4 Interpretation of Euler Equation

We conclude this section with a brief discussion of Levhari and Liviatan's (1977) interpretation of the Euler Equation (2.39), which may be rewritten:

$$e^{-rx} \left.\frac{\partial \Pi}{\partial y}\right]_x = \frac{d}{dx} \left[e^{-rx} \left.\frac{\partial \Pi}{\partial y'}\right]_x \right] . \quad (2.42)$$

Integrate (2.42) over the interval $[x, x_2]$, where x is arbitrary (but within $[0,x_2]$), and multiply by e^{rx}:

$$\int_x^{x_2} e^{-r(t-x)} \left.\frac{\partial \Pi}{\partial y}\right]_t dt = \int_x^{x_2} \frac{d}{dt} \left[e^{-r(t-x)} \left.\frac{\partial \Pi}{\partial y'}\right]_t \right] dt , \quad (2.43)$$

$$= \left[e^{-r(t-x)} \left.\frac{\partial \Pi}{\partial y'}\right]_t \right]_x^{x_2} ,$$

Table 4

THREE NUMERICAL SOLUTIONS FOR THE SHUT-DOWN DATE IN THE CASE OF A QUADRATIC/LINEAR COST FUNCTION

Parameters	Parameter Sets(a) I	II	III
K	\$104.50/ton	\$104.50/ton	\$104.50/ton
α	\$200,000.00/year	\$200,000.00/year	\$200,000.00/year
γ*	\$1.96/ton	\$1.96/ton	\$1.96/ton
θ	\$10.0/ton	\$10.0/ton	\$10.0/ton
r	0.07% per year	0.07% per year	0.07% per year
n	-0.05% per year	-0.02% per year	0.0001%per year
β	\$0.00001/ton²	\$0.00001/ton²	\$0.00643/ton²
Solution			
x_2	10.7 years	16.1 years	20 years

Notes: (a) The optimal extraction paths correspond to Panels I, II and III (respectively) of Figure 1.

(iii) As is suggested by the near-quadratic form (displayed in Table 3) of the necessary condition in x_2 (obtained by equating (3.28) and (3.29b)), in some cases there are two roots for x_2, only one of which is valid. The valid root must be selected by checking that the rest of the first-order conditions (as set out in Table 2) are satisfied.

(iv) When computing a contemporaneous comparative static elasticity $\eta(t)$, it must be checked that x_2 exceeds t (if it does not, then $\eta(t)$ is not defined).

A computer program was developed to find valid solutions for x_2 using any chosen values for K, α, γ^*, θ, r, n and β. For example, the three parameter sets listed in Table 4, which possess, respectively, the optimal extraction rate paths corresponding to Panels I, II and III of Figure 1, were verified using the computer program. Parameter set III of Table 4 was used to illustrate the components of $y'(x)$ plotted in Figure 2.

A second computer program was designed to complete grid searches, again testing for valid solutions of x_2, using all possible combinations of parameter values selected from multiple parameter sets. Two such multiple sets are reproduced in Table 5. There are a total of 729 possible parameter combinations which may be selected from each of them.

Table 2

PRINCIPAL RESULTS IN THE THEORY OF OPTIMAL EXTRACTION OF A MINERAL DEPOSIT(a),(b)

Planning Horizon when x_2 is:	When Reserves are: Abundant	Scarce
Unknown, to be chosen optimally	Total profit, F, at shut-down, = 0. (See Equation (2.20) with shadow price λ set = 0.) Marginal profit flow at closure point, $[\partial F/\partial y' \mid x = x_2]$ is zero. (See Eqn (2.37) with λ=0.) Average profit, F/y', is also zero at closure.	Total profit F at shut-down = (marginal profit at shut-down) × (rate of extraction at shut-down) and hence average profit at shut-down = marginal profit. (See Eqns (2.40) & (2.41a&b).) The shadow price of reserves = PV of marginal profit of last unit extracted (Eqn (2.37)).
Known, x_2 = T (say)(c)	The marginal profit at the end point T, $[\partial F/\partial y']_T$, is zero.	The marginal profit at the end-point T equals the shadow price of reserves: $\lambda = [\partial F/\partial y']_T > 0$.

Source: Based on Levhari and Liviatan (1977).

(a) The Euler Equation (2.39) holds in all cases.

(b) In this table the word 'profit' is used as a shorthand description for F, which strictly is the present value of profit.

(c) To obtain these results, delete from the discussion the following equations: (2.17b), (2.20), (2.40), (2.41a), (2.41b).

(2.44) $$= e^{-r(x_2 - x)} M\Pi(x_2) - M\Pi(x) \quad ,$$

where $M\Pi(\cdot)$ indicates the marginal profitability of extraction at $(\cdot)$. Consider now the left of (2.43). Since there is no reason to suppose that revenue, R, depends on cumulative extractions y, $\partial\Pi/\partial y$ may be replaced with $-\partial C/\partial y$, where C is total cost. Thus the left of (2.43) may be interpreted as minus the cumulated cost disabilities from x to the end of the plan of an additional unit of (cumulative) extraction at x. In Hotelling (1931), $\partial C/\partial y \equiv 0$, and (2.44) leads to the famous result that on an optimal plan, marginal profit must grow at the rate of discount. The first right hand term of (2.44) can be written:

(2.45) $$e^{-r(x_2-x)} M\Pi(x_2) = PV_x \{M\Pi(x_2)\} \; ;$$

that is, "the present value at x of the marginal profit at the end of the plan". Rearranging (2.44) we have:

(2.46) $$M\Pi(x) = PV_x \{M\Pi(x_2)\} + PV_x \{CCD(x,x_2)\} \quad ,$$

where the last term of (2.46) is just the negative of the left-hand side of (2.44), and is to be interpreted as "the present value at x of the cumulated cost disability over the remainder of the plan caused by the extraction of an additional unit at x". Splitting marginal profit into revenue MR and marginal cost MC, (2.46) may be rewritten:

(2.47) $$MR(x) = MC(x) + PV_x \{M\Pi(x_2)\} + PV_x \{CCD(x,x_2)\} = FMC(x) \text{ (say)},$$

where FMC is "full marginal cost", which is the sum of three components:

(a) ordinary short-run marginal cost at a fixed level of cumulative extractions;

(b) the opportunity cost of not waiting till the end of the plan to mine the last unit extracted in the current period;

(c) the future cost penalties incurred by mining the last unit in the current period.

4.3 Establishing a Parameter Set

In order to evaluate the quadratic/linear cost function and to solve for x_2, we must have information on the relevant interest rate for discounting, r; on the cost function parameters α, β and θ, γ^*; and finally on the initial product price K and on its expected rate of growth, n. The term α is fixed costs, the terms β and θ relate to current extraction costs at any given level of cumulative extractions, whilst γ^* represents the upward shift in the cost function for each additional unit of cumulative extractions (see (3.3)).

The selection of the nominal growth rates n and r imply a time unit of measurement for x. We evaluate n and r such that x is measured in years. We are not interested in the case of $n \geq r$ as this implies that the most profitable use of a resource is to leave it unmined, appreciating in value. The value of K (i.e., the initial price p(0)) is used as a numeraire: solutions depend only on costs relative to price. Thus solutions are homogenous of degree zero in (α/K, β/K, θ/K, γ^*/K).

Whilst, hopefully, our quadratic/linear cost function has application to Australian extractive industries, it is not our objective here to attempt any econometric estimations using empirical data. Rather, we seek first to establish how extensive the range of parameter values are which will provide valid solutions in the computation of x_2.

The criteria for validity of a solution are as follows:

(i) Output y(x) must be non-negative at all instants x in the plan ($x \in [0, x_2]$). (This is not guaranteed using the classical methods described above in Section 3.)

(ii) In addition to the first-order conditions identified in Table 2, appropriate second-order conditions must be satisfied (since the first-order conditions alone sometimes identify a saddle point; i.e., a maximum with respect to the form of y(x) but a minimum with respect to the shut-down date x_2 -- see the Appendix).

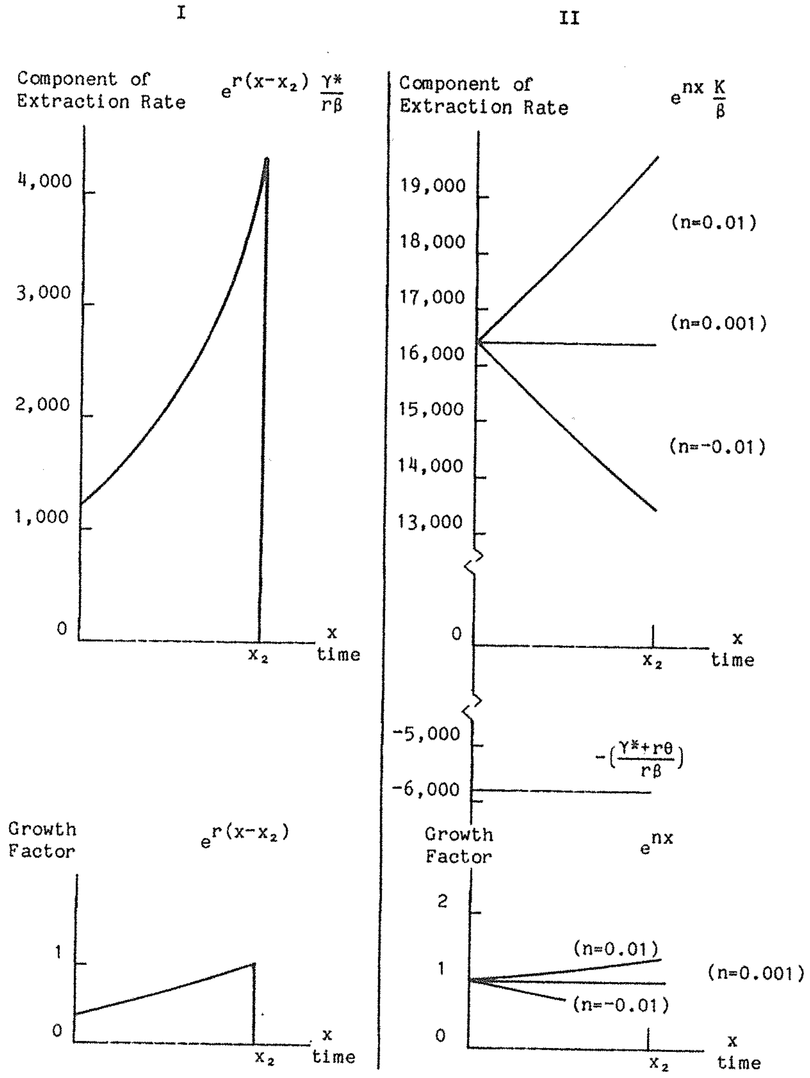

Figure 2 Components of the Optimal Extraction Path

We have already noted that (c) goes to zero for a mine which does not run into increasing costs as reserves are extracted. In the case where there is no shortage of reserves, and the shadow price of a unit of ore of the grade and accessability currently being extracted is zero, item (b) vanishes. Finally, if there is neither a shortage of reserves, nor any intertemporal externality in the cost function, the Euler Equation reduces to the ordinary result from static theory that MC(x) = MR(x) at every x.

3. SOLUTION OF THE MODEL FOR A QUADRATIC/LINEAR COST FUNCTION

3.1 Objectives

The terms of reference of this section are as follows:

Objectives: (a) To find the optimal time path of the extraction rate for a non-renewable resource when the price of the mineral is expected to grow at a uniform compound rate of n per cent per annum, and average extraction costs rise as a function of depletion.

(b) To find the elasticity of output (viz., of the extraction rate) at an arbitrary future date x, with respect to a sustained 1 per cent rise in expected product price, as a function of the variables and parameters in the model.

Remark: As mining operations proceed, it is likely that current costs will be greater, simply because previous mining has taken place. This could be because the mine is deeper; perhaps tailings need to be taken and dumped further away from the mine. The ore body may not be of uniform grade; the richer lodes will be exploited first leading to lower outputs of extracted minerals per real dollar of costs as depletion of the deposit proceeds.

3.2 Specific Assumptions

We continue with the convention $x_1=0$. The specific assumptions to be made in what follows are:

(3.1) $p(x) = Ke^{nx}$,

(the price p of the mineral is expected to grow at the uniform rate n, starting from an initial value of $p(0) = K$);

(3.2) $\Pi(x,y,y') = p(x)\, y'(x) - C(y'(x), y(x))$,

where

(3.3) $C(y'(x), y(x)) = \alpha + \theta y'(x) + \frac{1}{2}\beta\,[y'(x)]^2 + \gamma^* y(x)$,
$(\alpha,\beta,\gamma^*,\theta \geq 0)$.

In (3.2), $p(x)\, y'(x)$ is gross revenue flow at time x, and C is cost flow at that time. Equation (3.3) makes it clear that there are fixed costs α; and that when the rate of extraction y' increases, costs rise, with marginal costs varying linearly with y' at fixed y. Also, the rate of total cost flow rises linearly with the extent y of existing depletion ($\gamma^* > 0$).

3.3 A Particular Solution

The right-hand side of the Euler Condition (2.39) in the present case is:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left\{e^{-rx}\frac{\partial \Pi}{\partial y'}\right\}$$

$$= \frac{d}{dx}\left\{e^{-rx}\left[p(x) - \left.\frac{\partial C}{\partial y'}\right|_x\right]\right\}$$

$$= \frac{d}{dx}\left\{e^{-rx}[p(x) - \beta y'(x) - \theta]\right\}$$

$$= e^{-rx}\left\{p'(x) - \beta y''(x)\right\} - re^{-rx}[p(x) - \beta y'(x) - \theta]$$

The optimal extraction path implied by (4.1) may be I: of decline, II: decline then growth, or III: growth. It is not possible, given our quadratic/linear cost function, to have IV: a concave optimal extraction path. Figure 1 demonstrates these characteristics of equation (4.1).

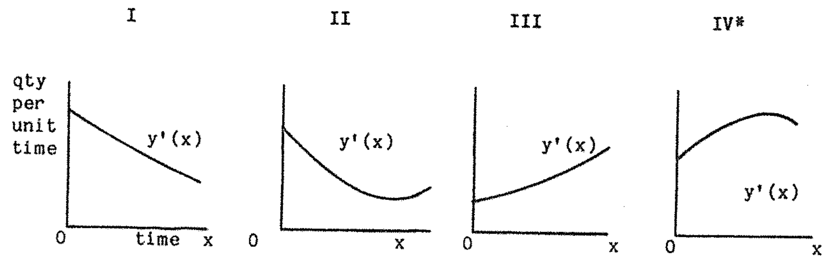

Figure 1 Optimal Extraction Paths: Three Solution Paths and One Non-Solution Path I, II and III are solution paths to equation (3.33). IV* is not a solution.

The first term on the right of equation (4.1), which is unambiguously positive, is composed of a growth variable $e^{r(x-x_2)}$, and a coefficient $\gamma^*/(r\beta)$. The entire term reaches a maximum value of $\gamma^*/(r\beta)$ when $x = x_2$. One possible path for the first term in equation (4.1) is plotted in Panel I of Figure 2. The second term, $e^{nx} K/\beta$, also is ambiguously positive, but it may either grow ($n > 0$) or decline ($n < 0$). The initial value of the term (at $x = 0$) is K/β. When n is negative the declining growth path demonstrated in Panel I of Figure 1 is possible. However, the positive growth effects of r may eventually overtake that of negative growth (due to $n < 0$), with the optimal extraction path declining and then growing (see Panel II, Figure 1). The positive growth due to the first term may dominate when $n < 0$, even from time $x = 0$. In this latter case, Panel III of Figure 1 would be the outcome. Finally, it is noted that as all of the elements of the third term in equation (4.1) are positive, the entire term remains negative.

(3.37b) $\Psi = p(x_2)\left[\frac{p(x_2)}{\gamma^*} - \frac{\theta}{\gamma^*} - \frac{1}{n}\right]$.

Once x_2 is found, Ψ and $\partial x_2/\partial K$ can be computed. At that stage, (3.34) can be used to compute $\partial y'(x)/\partial K$ and hence $\partial[\ln y'(x)]/\partial[\ln K]$.

4. SIMULATION RESULTS FOR THE QUADRATIC/LINEAR COST FUNCTION

4.1 Objectives

We now report our first efforts to implement the analytical solution set out in the proceeding section. We assume that x_2 is not known, and that reserves are abundant - that is $y(x_2) < a$. The terms of reference for this section are:

(a) to observe the range of shapes that optimal extraction paths may take (see (3.33)), given our quadratic-linear cost function (see (3.3));

(b) to evaluate equation (3.33), and to subject it to sensitivity analysis;

(c) to report the ORANI price elasticities of supply (see (3.32)) which are suggested by various arbitrary parameter sets.

Our analysis has been completed with the use of a micro-computer and software designed specifically to test for valid solutions to our quadratic/linear cost function.

4.2 Optimal Extraction Paths

Using (3.33) and (3.9') we obtain:

(4.1) $$y'(x) = e^{r(x-x_2)}\frac{\gamma^*}{(r\beta)} + e^{nx}\frac{K}{\beta} - \left(\frac{\gamma^*+r\theta}{r\beta}\right) ,$$

where

$$K,\ r,\ \beta > 0;\quad x_2,\ x,\ \gamma^*,\ \theta \geq 0;\quad x \leq x_2.$$

$$= e^{-rx}\{[p'(x) - r\,p(x)] - \beta[y''(x) - ry'(x)] + r\theta\}$$

(3.4) $$= e^{-rx}[P(x) - \beta Y(x) + r\theta] \quad \text{(say)}.$$

Above, $P(x)$ and $Y(x)$, respectively, are defined by:

(3.5a) $P(x) = p'(x) - r\,p(x)$;

(3.5b) $Y(x) = y''(x) - r\,y'(x)$.

The left-hand side of the Euler Equation (2.39) is simply:

$$\frac{\partial F}{\partial y} = e^{-rx}\frac{\partial \Pi}{\partial y}$$

$$= -e^{-rx}\frac{\partial C}{\partial y}$$

(3.6) $$= -\gamma^* e^{-rx} .$$

Equating (3.4) and (3.6), we obtain as the Euler Equation in this special case:

(3.7) $$\gamma^* = \beta Y(x) - P(x) - r\theta.$$

From (3.7) we see that given a time-path of prices, as encapsulated in $\{P(x): x \in [0, x_2]\}$, the path of the extraction-related variable $Y(x)$ is also known from (3.7). It is clear that $Y(0) = 0$, but $Y(x_2)$ and x_2 have to be worked out from the necessary conditions given in the upper half of Table 2.

From assumption (3.1) we see that $p'(x) = np(x)$, and hence from (3.5) that:

(3.8) $$P(x) = n\,p(x) - r\,p(x)$$

$$= (n-r)\,Ke^{nx} .$$

Substituting from (3.8) into (3.7), and rearranging, we obtain:

(3.9) $$Y(x) = \frac{\gamma}{\beta} + \frac{(n-r)}{\beta}Ke^{nx},$$

where

(3.9') $\Upsilon = \Upsilon^* + r\Theta$.

This says $[\Upsilon - (\Upsilon/\beta)]$ grows exponentially at the rate n. A particular solution of (3.9) is found by noting that (3.9) is compatible with $(y' + \frac{\Upsilon}{\beta r})$ growing exponentially at the rate n. Proof: Let

(3.10) $$y'(x) + \frac{\Upsilon}{\beta r} = Ae^{nx} .$$

Then

(3.11) $$y''(x) = n\,Ae^{nx}.$$

Substitute from (3.10) and (3.11) into (3.5b):

(3.12) $$Y(x) = n\,Ae^{nx} - r[Ae^{nx} - \Upsilon/(\beta r)];$$

that is,

(3.13) $$y''(x) - r\,y'(x) = \frac{\Upsilon}{\beta} + (n - r)\,Ae^{nx}.$$

Comparing (3.9) and (3.13), we see that $A = K/\beta$. Integrating (3.10) with $A = K/\beta$, we see that:

(3.14) $$y(x) = \frac{Ke^{nx}}{(n\beta)} - \frac{\Upsilon x}{(r\beta)} + const_1$$

is a particular solution for y.

3.4 General Solution

The second-order linear differential equation formed from the homogeneous part of (3.13), whose characteristic polynomial (a quadratic) is:

(3.15) $$1.z^2 - rz = 0,$$

has roots r and zero. Accordingly, the general solution for the homogeneous part of (3.13) is (see, e.g., Cogan and Norman (1958, p. 207)):

(3.16) $$y(x) = const_2\, e^{rx} + const_3 .$$

Adding the particular solution (3.14) to the general solution (3.16) of the homogeneous part of (3.13), we find (3.13)'s general solution to be:

In a long-run closure with a given chosen value of t (10 years, say), the price elasticity of supply of a miner facing the price expectations specified in (3.1), and the technology specified in (3.3), is

(3.32) $$\frac{\partial[\ln y'(t)]}{\partial[\ln\ p(t)]} = n(t)\ \ (say) = \frac{\partial[\ln y'(t)]}{\partial[\ \ln K]} \quad (all\ t) .$$

Differentiating (3.27) with respect to x, we obtain

(3.33) $$y'(x) = e^{r(x-x_2)}\frac{\Upsilon^*}{(r\beta)} + \frac{K}{\beta}e^{nx} - \frac{\Upsilon}{(r\beta)} .$$

With x fixed ($x \in [0, x_2]$), differentiate (3.33) with respect to K (whilst keeping in mind that x_2 is a function of K):

(3.34) $$\frac{\partial y'(x)}{\partial K} = -\frac{\Upsilon^*}{\beta}e^{r(x-x_2)}\frac{\partial x_2}{\partial K} + \frac{e^{nx}}{\beta} .$$

We now develop an expression for $\partial x_2/\partial K$ from the listing in Table 3 of the coefficients of variables involving x_2 in the equation obtained by equating (3.28) and (3.29). We see that

$$A\frac{\partial}{\partial K}(e^{2nx_2}) + e^{2nx_2}\frac{\partial A}{\partial K} + B\frac{\partial}{\partial K}(e^{nx_2}) + e^{nx_2}\frac{\partial B}{\partial K}$$

(3.35) $$+ C\frac{\partial}{\partial K}(e^{-rx_2}) + e^{-rx_2}\frac{\partial C}{\partial K} + D\frac{\partial x_2}{\partial K} + \frac{\partial E}{\partial K} = 0 .$$

From Table 2 we see that $\partial C/\partial K = 0$, while

(3.36a) $$\partial A/\partial K = K/(\beta\Upsilon^*) ,$$

(3.36b) $$\partial B/\partial K = -\frac{1}{\beta}\left(\frac{\Theta}{\Upsilon^*} + \frac{1}{n}\right) ,$$

and

(3.36c) $$\partial E/\partial K = 1/(n\beta) .$$

If follows that

(3.37a) $$\partial x_2/\partial K = -\left[\frac{1}{n} + \frac{\Psi}{K}\right] / \left[n\Psi - \frac{\Upsilon^* e^{-rx_2}}{r} + \frac{\Upsilon}{r}\right] ,$$

where

Table 3
COEFFICIENTS OF VARIABLES INVOLVING THE SHUT-DOWN DATE, x_2, IN THE EQUATION IMPLICITLY DETERMINING x_2

Variable		Coefficient
e^{2nx_2}	A =	$K^2 / [2\beta\gamma^*]$
e^{nx_2}	B =	$-\frac{K}{\beta}(\frac{\theta}{\gamma^*}+\frac{1}{n})$
e^{-rx_2}	C =	$\frac{\gamma^*}{(r^2\beta)}$
x_2	D =	$\gamma/(r\beta)$
constant	E =	$\frac{\theta^2}{2\beta\gamma^*} - \frac{\alpha}{\gamma^*} - \frac{\gamma^*}{(r^2\beta)} + \frac{K}{(n\beta)}$

3.6 <u>The Elasticity of Mining Output with Respect to a Sustained Price Rise</u>

In ORANI, the form of shock imposed in a simulation is as follows: Suppose that, starting now, a certain exogenous variable were to deviate from its control path by z per cent. After t years, how much (in percentage terms) would some particular endogenous variable differ from its control path? In short-run simulations, t is about 2 years. In long-run simulations, the user may choose t; for the assumption underlying the model to be appropriate, however, t probably should not be chosen less than 5 years in the case of most shocks.

Consider now a 1 per cent sustained shock in the expected price of the mineral. Then at a given future date t, the proportional change in the expected price p(t) (see (3.1)) is

(3.31) $d \ln p(t) = d \ln K = .01$

(3.17a) $y(x) = const_2\, e^{rx} + \frac{K}{(n\beta)} e^{nx} - \frac{\gamma x}{(r\beta)} + (const_1 + const_3)$.

Renaming constants, (3.17a) is:

(3.17b) $y(x) = C_1\, e^{rx} + \frac{K}{(n\beta)} e^{nx} - \frac{\gamma x}{(r\beta)} + C_2$.

[To check (3.17b), take its first and second derivatives, then form Y according to definition (3.5b), and verify that (3.9) is recovered.] To evaluate C_1 and C_2, we make use of the end-point conditions in the upper half of Table 1. For many of Australia's most important mineral exports -- coal, bauxite, iron ore -- the more interesting case in Table 2 is the one in which reserves are abundant. The relevant conditions are:

(3.18) $F(x_2,[y(x_2)],[y'(x_2)]) = 0;$

(3.19) $\left.\frac{\partial F}{\partial y'}\right|_{x_2} = 0$.

Working first with (3.18), via (2.2) and (3.2) we have:

$$\Pi\big|_{x_2} = p(x_2)\, y'(x_2) - C(y'(x_2),\, y(x_2)) = 0 ;$$
$$= p(x_2)\, y'(x_2) - \alpha - \theta y'(x) - \tfrac{1}{2}\beta\, [y'(x_2)]^2 - \gamma^* y(x_2);$$

that is,

(3.20) $K e^{nx_2}\, y'(x_2) = \alpha + \theta y'(x) + \tfrac{1}{2}\beta\, [y'(x_2)]^2 + \gamma^* y(x_2)$.

Turning now to (3.19), we obtain [see the first three lines of (3.4)]:

(3.21) $\left.\frac{\partial F}{\partial y'}\right|_{x_2} - e^{-rx_2}\left(K e^{nx_2} - \beta\, y'(x_2) - \theta\right) = 0$;

so

(3.22) $y'(x_2) = \frac{K}{\beta} e^{nx_2} - \frac{\theta}{\beta}$.

But from (3.17b),

(3.23) $$y'(x_2) = r\,C_1 e^{rx_2} + \frac{K}{\beta} e^{nx_2} - \frac{\Upsilon}{r\beta} \;.$$

Equating (3.22) and (3.23) we see that

(3.24) $$C_1 = e^{-rx_2}\left[\frac{\Upsilon^*}{(r^2\beta)}\right] \;.$$

Using the initial condition $y(0) = 0$, from (3.17b) we see that

(3.25) $$-(C_1 + C_2) = \frac{K}{n\beta} \;;$$

thus from (3.24),

(3.26) $$C_2 = -\left\{\frac{K}{(n\beta)} + e^{-rx_2}\left[\frac{\Upsilon^*}{(r^2\beta)}\right]\right\} \;.$$

Substituting from (3.24) and (3.26) into (3.17b), the solution for cumulative extractions $y(x)$ is:

(3.27) $$y(x) = e^{r(x-x_2)}\left[\frac{\Upsilon^*}{(r^2\beta)}\right] + \frac{K}{(n\beta)} e^{nx} - \frac{\Upsilon x}{(r\beta)} - \left[\frac{K}{(n\beta)} + e^{-rx_2}\left(\frac{\Upsilon^*}{(r^2\beta)}\right)\right] \;.$$

3.5 The Optimal Planning Horizon

To this point, with the exception of determining the value of x_2, we have satisfied objective (a) of this section under assumptions (3.1) and (3.3). We now derive an equation which implicitly determines x_2. This also serves as a preliminary to pursuing objective (b).

If we put $x = x_2$ in (3.27) we obtain:

(3.28) $$y(x_2) = \frac{1}{\beta}\left[\frac{K}{n}\{e^{nx_2} - 1\} - \frac{\Upsilon^*}{r^2}\{e^{-rx_2} - 1\} - \frac{\Upsilon x_2}{r}\right].$$

Now from (3.20),

$$y(x_2) = \frac{1}{\Upsilon^*}\{Ke^{nx_2}\, y'(x_2) - \theta y'(x_2) - \alpha - \tfrac{1}{2}\beta[y'(x_2)]^2\} \;;$$

(by substitution from (3.22))

(3.29a) $$= \frac{1}{\Upsilon^*}\left[\frac{\beta}{2}\{y'(x_2)\}^2 - \alpha\right]$$

(3.29b) $$= \frac{1}{\Upsilon^*}\left[\frac{Ke^{nx_2}}{\beta}\left\{-\theta + \frac{Ke^{nx_2}}{2}\right\} + \frac{\theta^2}{2\beta} - \alpha\right] \;.$$

Equating (3.28) with (3.29b) we obtain an expression which, in principle, determines x_2. It is clear that such an expression involves the first, second, and $(-r/n)$th powers of e^{nx_2}, as well as x_2. We cannot, therefore, obtain a closed-form solution for x_2. For future use, we note that the equation obtained from (3.28) and (3.29), written with zero on the right, has coefficients as set out in Table 3. The roots of this equation should be relatively easily found with the aid of a non-linear equation package.

An alternative is to construct, and then refine, an approximate solution obtained by taking Taylor expansions of e^{nx_2}, e^{2nx_2}, and e^{-rx_2}, about a guess X of x_2, while dropping terms of third and higher order. The relevant approximations are:

(3.30a) $$e^{nx_2} = e^{nX} e^{n(x_2-X)} \cong e^{nX}\left[1 + n(x_2-X) + n^2\frac{(x_2-X)^2}{2}\right] \;;$$

(3.30b) $$e^{2nx_2} = e^{2n} e^{2n(x_2-X)} \cong e^{2nX}\left[1 + 2n(x_2-X) + 2n^2(x_2-X)^2\right] \;;$$

and

(3.30c) $$e^{-rx_2} = e^{-rX} e^{-r(x_2-X)} \cong e^{-rX}\left[1 - r(x_2-X) + r^2\frac{(x_2-X)^2}{2}\right] \;.$$

Use of these equations, and Table 3, leads to a quadratic in x_2.