IMPACT OF DEMOGRAPHIC CHANGE ON INDUSTRY STRUCTURE IN AUSTRALIA

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A DYNAMIC MODEL OF THE FIRM-HOUSEHOLD WITH AN IMPERFECT CAPITAL MARKET

by

Keith R. McLaren
Monash University

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IMPACT Project, Industries Assistance Commission, 608 St. Kilda Road, Melbourne, Victoria 3004 Australia
Area Code 03 Telephone 51 8611

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1. Introduction.

In the existing literature, the two major areas of application of dynamic models, apart from optimal growth, are the theory of investment and the theory of consumption. The majority of investment models take as given that the criterion function is the present value of the firm. This assumption can be traced back to Fisher (1930), who pointed out that, under the assumption of a perfect capital market, consumption and investment decisions can be separated. The firm acts to maximize its present value, and the consumer takes this present value as given to maximize utility. In the investment literature we should mention Arrow (1968a), Jorgenson (1967), Eisner and Strotz (1963), Treadway (1969, 1970), Lucas (1967) and Gould (1968), and a variation of the criterion function by Wong (1975). In consumption theory, there are a number of theoretical models, such as Tintner (1938), and recently Lluch (1973a, 1973b, 1974) has developed an empirically implementable model of intertemporal consumption theory, called the Extended Linear Expenditure System (E.L.E.S.). Essentially Lluch takes the present value of the household as given, in the Fisherian tradition.

If we are interested in a joint firm household entity, such as a farm, where a single decision making unit carries out decisions on consumption, production, investment and financing, the initial temptation is to appeal to

Fisher's suggestion, treat the productive entity first to maximize present value, use this present value to determine consumption behaviour, and hence to simply combine the above models. But the assumption of a perfect capital market, a requisite of such an approach, appears to be a critical violation of reality. The approach taken in this paper is to assume an imperfect capital market, and treat the consumption-investment-production-financing decisions as interdependent. A simple application of this method to the theory of the firm is given in McLaren (1976).

The framework of Lluch's E.L.E.S. is taken as the basic starting point, and E.L.E.S. is reformulated in a way to allow further extensions in Section 2, with Lluch's basic results presented as equations (2.12) to (2.14). It is here that we introduce the idea of synthesizing a closed loop control and discuss its econometric implications, which is not considered by Lluch. In Section 3 the E.L.E.S. framework is reformulated to allow production and investment decisions. In essence, all that is required is to identify non-human wealth with capital stock and the rate of interest with marginal productivity of capital (given variable factors are optimally adjusted), and similar results follow provided the production function is linearly homogeneous. Thus capital goods have replaced the perfect capital market. Section 4 reintroduces the financial capital market, and considers the way in which an imperfect financial capital market can be integrated into the model. The average rate of interest is assumed to be an increasing function of the debt asset ratio.

The simple examples of Sections 3 and 4 are expanded into a more realistic model in Section 5, where duality theory is used to simplify the structure of the model by separating the atemporal and intertemporal optimization problems. Econometric problems are considered in Sections 6 and 7.

2. The Extended Linear Expenditure System.

The Extended Linear Expenditure System is introduced by Lluch (1973a) and further developed in a number of other papers (1973b, 1974). This model may be clarified and made more accessible by casting it within a framework more consistent with modern control theory (see, for example, Pontryagin et al (1962), Intriligator (1971), Hestenes (1966)). Such a reformulation may also open the way for further generalization.

Some (admittedly rather minor) problems with Lluch's derivation in Lluch (1973a) are the following. First, there is no explicit identification of the state and control variables required to reduce the model to a form in which modern control theory results may be applied. Second, the budget constraint as initially stated does not seem to be properly formulated, serving only to define the variable non-human wealth in planning time t. However, footnote 9, p.25 of Lluch (1973a) shows a transversality condition has been applied, which effectively converts this definitional equation into a binding budget constraint. For similar comments on the appropriate budget constraint, see Arrow (1969). Third, the domain of functions over which the search for an optimum is to take place is not explicitly specified. For example, equation 5 on p.26 of Lluch (1973a) suggests that q is to be continuously differentiable. In fact the class of admissible functions can be rather wider than this. Finally, Lluch concentrates on behaviour at time $t = 0$ when planning and historical time coincide. Operationally, this is all that is required if continual replanning is assumed. However, it is of some interest to note that E.L.E.S. is an example of a control problem in which synthesis is possible; that is, a closed loop feed-back control law expressing the optimal control at each point in time as a function of the state at that time can be derived explicitly. Such "well-behaved" control functions seem to be very rare, and the majority of control theory references quote only the quadratic criterion, linear control law case as an example in which synthesis is possible. (A well known example from economics which falls into this category is the Eisner-Strotz (1963) investment function). Econometrically, closed loop control relations are important because it is these relations which are stable across samples and through time. Incidentally, closed-loop control laws ensure consistency in the sense of Strotz (1956), which is not so obvious in the case of "initial period" controls. In the case of E.L.E.S., however, inconsistency is not a problem, by a result due to Strotz.

Following Lluch, define the following set of variables:

$q(t)$	:	n - vector of consumption good flows, $q \geqslant 0$.
P	:	n - vector of prices corresponding to q .
$y(t)$	:	exogenous flow of labour income .
M	:	initial money holdings .
δ	:	subjective rate of time discount .
ρ	:	rate of interest in a perfect capital market .
U	:	instantaneous utility function, defined on q .
E	:	total consumption expenditure = $P'q$.

The function U is assumed continuously differentiable on some domain $R \subset E^n$.

At time $t = 0$, we consider the consumers problem as that of choosing $q(t)$, $0 \leqslant t \leqslant \infty$, from among the class of piecewise continuous functions q taking values in R, to maximize

$$(2.1) \qquad J = \int_0^\infty e^{-\delta t} U[q(t)]dt$$

subject to an appropriate budget constraint. In a world of perfect capital markets, we can imagine the possibility of capitalizing all future income and allocating this among future consumption with appropriate prices, $P(t)e^{-\rho t}$. Then one possible budget constraint is

$$(2.2) \qquad \int_0^\infty e^{-\rho t} P(t)' \, q(t) \; dt \leqslant \int_0^\infty e^{-\rho t} \, y(t) \; dt + M .$$

(2.2) is the form of an isoperimetric constraint, and maximizing (2.1) subject to (2.2), with U a Klein-Rubin utility function, generates E.L.E.S. (The appropriate necessary conditions are given by Theorem 5.1, p.263 of Hestenes (1966)). This is a constrained problem in the calculus of variations, since states, controls and a control law have not been identified, $q(t)$ must be assumed piecewise smooth. The alternative approach explored below is preferred because the state variable is made explicit.

With a view to allowing for continuous replanning, introduce the variable $W(t)$ defined as net worth of the consumer as evaluated at time t. More explicitly, given the path $y(t)$, define $W(o)$ by

$$(2.3) \qquad W(o) = \int_0^\infty e^{-\rho t} \, y(t) \; dt + M$$

since with a perfect capital market the consumer could "sell" his future income stream $y(t)$ for $W(o) - M$. Now for any path $q(t)$ define $W(t)$ by

$$(2.4) \qquad \dot{W}(t) = \rho W(t) - P'q(t)$$

and the initial condition (2.3). The following are equivalent definitions:

$$(2.5) \qquad W(t) = W(o) + \int_o^t (\rho W(s) - P'q(s))ds$$

$$(2.6) \qquad W(t) = Me^{\rho t} + \int_t^\infty e^{-\rho(s-t)} y(s)\, ds + \int_o^t e^{\rho(t-s)} (y(s) - P'q(s))ds .$$

Because of these relations, it is argued that $W(t)$ is the appropriate state variable. If replanning is considered at time t, all relevant information is contained in $W(t)$. The appropriate control variables are q, and the system (2.1), (2.3) and (2.4) is in the form of a standard control problem. Note that W is not Lluch's w.

So far we have merely formulated the problem in control theory language, and a right hand end-point condition is yet to be stated. The economic problem is introduced by the budget constraint, which in this case must be

$$(2.7) \qquad W(t) \geqslant 0 , \quad 0 \leqslant t \leqslant \infty$$

i.e. planned bankruptcy is not allowed. Note that (2.4) is a definition, (2.7) is the constraint. Now restricting attention to $P(t) \geqslant 0$,

$q(t) \geqslant 0$, it can be seen from (2.4) that if $W(t)$ were to become negative at some finite t, say t_o, then this would imply

$$W(t) < 0 \ , \ t_o \leqslant t \leqslant \infty$$

Thus the following constraint,

$$\text{(2.8)} \qquad \lim_{t\to\infty} W(t) \geqslant 0$$

when taken together with (2.4) and $P,q \geqslant 0$, is equivalent to the constraint (2.7). Thus the budget constraint (2.8) gives the required right hand end point condition.

The following control problem can now be formulated:

For the 1-dimensional state variable $W(t)$ and the n-dimensional control vector $q(t)$, and the control law

$$\dot{W} = \rho W - P'q$$

find the trajectory $W^*(t)$, $q^*(t)$ to maximize (2.1) subject to the initial condition (2.3), the end point condition (2.8), and any other restrictions on W,q. The class of admissible trajectories over which a maximum is sought is the class:

$$\beta = \{W(t), q(t) : W(t) \text{ piecewise smooth, } q(t) \text{ piecewise continuous, } q(t) \in R\} .$$

R can account for economic constraints, such as $q \geqslant 0$, and technical constraints, such as ensuring U continuously differentiable on its domain.

Apart from the fact that:

(a) integration is over an infinite interval, and

(b) the corresponding form of the right-hand end point condition (2.8), we are in the area of classical control theory, and necessary conditions are available in many references (for example, Pontryagin, et al. (1962), Hestenes (1966), Hadley and Kemp (1971)). The extensions required to include conditions (a) and (b) are to be found in Arrow (1968b). Constraints on controls, such as $q \geqslant 0$ (or $q \geqslant \gamma$) are easy to handle within this framework, but since the nature of the utility function is usually such as to lead to an interior solution as an optimality condition, we begin by assuming $q^*(t) > 0$.

Following Hestenes (1966) Theorem 2.1, p.254, we form

$$H(t,W;q;\Psi) = \lambda_o e^{-\delta t} U(q) + \Psi(t)\ (\rho W(t) - P'q(t)) .$$

and the optimal path W^*, q^* satisfies the following necessary conditions:

(i) $\lambda_o > 0$ and $\Psi(t)$ do not vanish simultaneously for any t.

$\Psi(t)$ is continuous on $[0,\infty]$ and

(ii) $\dot{W} = H_\Psi = \rho W - P'q$

(iii) $\dot{\Psi} = -H_W = -\rho\Psi(t)$.

(iv) $H_{q_i} = \lambda_o\ e^{-\delta t} U_{q_i} - P_i \Psi = 0 \quad i = 1, \ldots, n$.

and from Arrow (1968b) p.93 ,

(v) $\lim_{t\to\infty} \Psi(t) \geqslant 0 \qquad \lim_{t\to\infty} \Psi(t)\ W(t) = 0$

Providing that $U_{q_i} > 0$ it follows from (i) and (iv) that $\lambda_o > 0$, and so can be normalized at $\lambda_o = 1$. (If $U_{q_i} = 0$ at q^*, then q^* is an unconstrained maximum and (i), (v) give $\Psi = 0$, so again $\lambda_o = 1$).

From (iii),

(iiia) $\quad \Psi(t) = \Psi_o e^{-\rho t}$

and the set of necessary conditions is

$$\text{(ii)} \qquad \dot{W} = \rho W - P'q$$

$$\text{(iva)} \qquad U_{q_i} = P_i \Psi_o e^{(\delta-\rho)t}$$

$$\text{(va)} \qquad \lim_{t\to\infty} e^{-\rho t} W(t) = 0$$

the initial condition on $W(0)$.

Now if (iva) can be solved to give q_i as a function of Ψ_o (a constant) and t, substitution in (ii) gives a linear differential equation in W with two end point conditions. One of these is used to determine Ψ_o, and one to eliminate a constant of integration. Solution gives the open-loop state path $W^*(t)$, the value of Ψ_o, and hence the open loop control for $q^*(t)$. Note that the differential equation is linear regardless of the shape of U, provided only that (iva) can be solved for q, so that an explicit open-loop solution is always possible. Synthesis of q^* in terms of W^* is the difficult, and economically meaningful, problem.

Following Lluch, if $U(q)$ is specified to be of the Klein-Rubin form:

(2.9) $$U(q) = \beta' \ln (q-\gamma) \qquad \beta' 1 = 1$$

then

$$U_{q_i} = \frac{\beta_i}{q_i - \gamma_i}$$

and (iva) gives

$$P_i q_i = P_i \gamma_i + \frac{\beta_i}{\Psi_o} e^{(\rho-\delta)t}$$

Substituting in (ii) :

$$\dot{W} = \rho W - P'\gamma - \frac{e^{(\rho-\delta)t}}{\Psi_o}$$

which has a solution of the general form:

(2.10) $$W(t) = \frac{e^{(\rho-\delta)t}}{\Psi_o \delta} + \frac{P'\gamma}{\rho} + C_1 e^{\rho t} ,$$

C_1 a constant of integration (see, for example, Boyce and Di Prima (1969) p.13). The transversality condition (va) gives $C_1 = 0$, and then the initial condition gives

$$\frac{1}{\Psi_o} = \delta W(o) - \frac{\delta}{\rho} P'\gamma$$

Thus the open-loop state $W^*(t)$ is determined. With $C_1 = 0$ (which used the explicit solution for Ψ) all we need now is

$$\frac{e^{-\delta t}}{\Psi(t)} = \delta W(t) - \frac{\delta}{\rho} P'\gamma$$

to give

(2.11) $$\dot{W} = (\rho-\delta) W(t) + P'\gamma \left(\frac{\delta}{\rho} - 1\right)$$

$$= (\rho-\delta) \left(W(t) - \frac{P'\gamma}{\rho}\right)$$

relating growth or decline of $W(t)$ to $\rho \gtrless \delta$. Another state variable of interest may be "supernumerary wealth" $V(t) = W(t) - \frac{P'\gamma}{\rho}$: $\dot{V} = (\rho-\delta)\, V$.

The open loop control q^* is also now available, and in this case synthesis is simple:

$$(2.12) \quad P_i q_i(t) = P_i \gamma_i + \beta_i \,\delta(W(t) - \frac{P'\gamma}{\rho})$$

$$(2.13) \quad E(t) = P'\gamma + \delta(W(t) - \frac{P'\gamma}{\rho})$$

Defining $z = \rho W$ as permanent income, we have Lluch's form of E.L.E.S.:

$$(2.14) \quad \hat{P}q = \hat{P}\gamma + \mu\beta(z - P'\gamma)\,, \quad \mu = \frac{\delta}{\rho} \;;$$

and all of Lluch's conclusions follow. The role of the perfect financial capital market has been to remove the dependence of the time path of $q(t)$ on the time path of $y(t)$. In the extreme imperfect capital market case, we would have $E(t) = y(t)$, and there are various alternatives of partial dependence in between.

3. The Introduction of Production and Investment

To introduce this generalization, consider the model of a household which owns an initial stock of a fixed capital good, say K_o, which is to be combined with a variable input, L, (labour) through a neoclassical production function to produce output $f(K,L)$ which is sold at a price s. Gross revenue $GR = sf(K,L)$ is used to pay the wages bill, wL, and the remainder is allocated between net investment, $c\dot{K}$, and consumption, $P'q$. c is the price of new investment goods. Thus there is no financial capital market, and capital goods take over the role of reproducing wealth and allocating it over time to consumption.

In this model, the state variable is $K(t)$, and the controls are $L(t)$ and the n-vector $q(t)$. The objective is to maximize

(3.1) $$J = \int_o^\infty e^{-\delta t} \; U[q(t)]dt$$

subject to the control law

(3.2) $$c\dot{K} = sf(K,L) - wL - P'q$$

an initial condition

$$K(o) = K_o$$

and a terminal condition $\lim_{t\to\infty} K(t) \geqslant 0$.

Therefore we set up the Hamiltonian

$$H(t, K; L, q; \psi) = e^{-\delta t} U(q) + \psi\left(\frac{s}{c} f(K,L) - \frac{w}{c} L - \frac{P'q}{c}\right)$$

and the necessary conditions for an interior solution are:

(i) $$c\dot{K} = sf(K,L) - wL - P'q$$

(ii) $$\dot{\psi} = H_K = -\psi\frac{s}{c} f_K(K,L)$$

(iii) $$H_{q_i} = e^{-\delta t}U_{q_i} - \psi \frac{P_i}{c} = 0 \qquad i=1, \ldots, n.$$

(iv) $$H_L = \psi\left(\frac{s}{c} \; f_L(K,L) - \frac{w}{c}\right) = 0$$

noting that λ_o has been set to unity without loss of generality, and static price expectations have been assumed.

In general, necessary conditions (i) - (iv), plus the end point conditions, will lead to a system of differential equations that is hopelessly non-linear, due to the non-linearities of the instaneous utility function and the production function. Thus a general analytical solution is probably impossible. However, there may exist a set of classes of instantaneous utility functions, say $\mathcal{U}$, a set of classes of production functions, say $\mathcal{P}$, and a subset, say $\mathcal{S}$ of $(\mathcal{U} \times \mathcal{P})$, such that each element of $\mathcal{S}$ allows explicit analytical solution of equations (i) to (iv) above, plus the end point conditions, in closed loop feedback form. $\mathcal{S}$ is not empty, since, for example,

$$U = \Sigma \beta_i q_i^2 \qquad \beta_i > 0$$

$$f(K,L) = aK + bL^2 \qquad a > 0 , b > 0$$

is a member of $\mathcal{S}$, (after introducing a dummy disposable consumption good to account for inequalities).

A complete characterization of $\mathcal{S}$ is desirable, but is not achieved here. However we do demonstrate a much more interesting member of $\mathcal{S}$, which may be useful for empirical work, and comment further on likely members of $\mathcal{S}$.

Consider the member of $\mathcal{S}$:

$$U(q) = \beta' \ell n\ (q-\gamma) \qquad \beta_i > 0 ,\ \beta' i = 1 ,\ \gamma_i \geqslant 0$$

$$f(K,L) = AK^{\alpha} L^{1-\alpha} \qquad 0 < \alpha < 1$$

i.e. a Klein-Rubin utility function and a constant returns-to-scale Cobb-Douglas production function. Then (iv) gives (iva) $$s(1-\alpha)\, AK^{\alpha}\, L^{-\alpha} = w$$

$$\text{i.e., } L = \left(\frac{s(1-\alpha)A}{w}\right)^{\frac{1}{\alpha}} K = aK \text{ , say .}$$

Note that the optimal capital-labour ratio is a function of w but not of c . (iva) can be used in (ii) to give

(iia) $$\begin{aligned} \dot{\psi} &= -\psi \frac{s}{c} \alpha A\, K^{\alpha-1}\, L^{1-\alpha} \\ &= -\psi \frac{s}{c} \alpha A\, a^{1-\alpha} \\ &= -\psi b \text{ , say. } \left(b = \frac{w}{c} \frac{\alpha}{1-\alpha} a\right) . \end{aligned}$$

Therefore $\psi(t) = \psi_o e^{-bt}$, ψ_o a constant, and

$$P_i q_i = P_i \gamma_i + \frac{c\beta_i}{\psi_o} e^{(b-\delta)t}$$

Finally, (iva) can be used to give

$$f(K,L) = A\, K^{\alpha}\, (aK)^{1-\alpha} = \frac{bc}{\alpha s} K$$

which together with the result for $P_i q_i$ and (iva) can be used in (i) to get the differential equation describing the state variable

(3.3) $$\dot{K} = bK - \frac{P'\gamma}{c} - \frac{e^{(b-\delta)t}}{\psi_o}$$

since $b/\alpha - wa/c = b$. (3.3) has a general solution of the form:

(3.4) $$K(t) = \frac{e^{(b-\delta)t}}{\psi_o\, \delta} + \frac{P'\gamma}{bc} + C_2\, e^{bt}$$

and transversality again gives $C_2 = 0$. Going directly to the closed loop feed back control law, we have

$$P_i q_i = P_i \gamma_i + \beta_i \delta (cK(t) - \frac{P'\gamma}{b})$$

$$= P_i \gamma_i + \beta_i \delta \frac{c}{w} (1-\alpha) (\frac{s(1-\alpha)A}{w})^{\frac{1}{\alpha}} \cdot (GR(t) - \frac{P'\gamma}{\alpha})$$

where $GR(t) = s\ f\ (K,L)$. The net investment function is

$$c\dot{K} = (b-\delta)\ (cK - \frac{P'\gamma}{b}) .$$

A more interesting observable variable may be net revenue, $N(t)$, defined as $s\ f(K,L) - wL = bcK$ along an optimal path. Then

$$c\dot{K} = (1- \frac{\delta}{b})\ (N(t) - P'\gamma)$$

$$P_i q_i = P_i \gamma_i + \frac{\beta_i \delta}{b}\ (N(t) - P'\gamma)$$

$$E = P'\gamma + \frac{\delta}{b}\ (N(t) - P'\gamma) .$$

Recall that $b = \frac{\alpha}{1-\alpha} \frac{w}{c} (\frac{s(1-\alpha)A}{w})^{\frac{1}{\alpha}}$, so that the effects of price changes (w, c, s) on consumption expenditure enter through this term.

A number of empirical formulations are now possible. A number of possible observable right-hand variables are: net worth $cK(t)$, gross revenue $GR = s\ f\ (K,L)$ or net revenue $N = s\ f(K,L) - wL$. As with L.E.S. and E.L.E.S., there is the choice of first estimating the expenditure function, and then the separate categories of expenditure, or starting directly from a whole-system estimation. These problems will not be discussed further here, but will be postponed until a more general model has been considered. Note, however, that GR and N are <u>not</u> exogenous in the sense in which K is.

The use of a constant-returns-to-scale Cobb-Douglas production function has allowed production, investment and labour demand to be included in an

E.L.E.S. type system. The homotheticity of a Cobb-Douglas ensures that along an optimal (planned) path, with static price expectations, the labour capital ratio is constant. The linear homogeneity of the function then gives a linear differential equation describing the state variable, which again can be solved explicitly. Thus it would seem that the set $\mathcal{S}$ contains all the linearly homogeneous production functions. This specification is sufficient to allow explicit open-loop solutions. An interesting question is whether there exist functions for which the closed-loop controls can be synthesized without explicit solution of open loop controls and state paths, for example, with non constant returns to scale production functions.

Note that derivations in this section carry through for a constant return to scale C.E.S. production function, with only the usual added algebraic complexity.

4. Introduction of Financing Decisions

In this section we extend the model of Section 3 to allow for the observed situation that many household-firm units operate on an overdraft. The model is equivalent to that in Section 3, except that there is now an additional source of funds, borrowing. The variable $B(t)$ represents total borrowing at time t, with an average rate of interest r. The constraint $B \geqslant 0$ will be imposed, since the rate of interest usually has a jump discontinuity at $B = 0$. Clearly borrowing to finance production is the more important case to consider. It turns out that a constant rate of interest is not the simplest case to consider, for this will lead to either a corner solution $(B=0)$ or an unbounded solution $(B=\infty)$, if the production function is constant returns to scale. All that we need to assume is that the rate of interest is a monotonic increasing function of either total debt, B, or the debt-capital ratio $Z = B/cK$. The latter is considered more realistic.

The new budget constraint is

$$(4.1) \qquad c\dot{K} = sf(K,L) - wL - P'q - rB + \dot{B} \; .$$

The state of the system at any point is given by the pair of state variables $K(t)$ and $B(t)$, from which we derive a third state, net worth

$$W = cK - B \; .$$

This fact will introduce a problem, since end point conditions will relâte to W. The controls are seen to be $q, L, \dot{B}$. To apply stantard control theory results, introduce the new control $A = \dot{B}$. Then we wish to maximize

$$(4.2) \qquad J = \int_0^\infty e^{-\delta t}\, U[q(t)]dt$$

subject to the control laws

$$(4.3) \qquad c\dot{K} = sf(K,L) - wL - p'q - rB + A$$

$$(4.4) \qquad \dot{B} = A \; .$$

In this case the Hamiltonian is

$$H(t, K, B; q, L, A; \psi_1, \psi_2)$$

$$= e^{-\delta t}\, U(q) + \frac{\psi_1}{c}\, (sf(K,L) - wL - P'q - rB + A) + \psi_2 A \; .$$

Necessary conditions for an interior maximum, using $r = r(Z)$, are

(i) $\dot{K} = H_{\psi_1}$ i.e., $c\dot{K} = sf(K,L) - wL - P'q - rB + A$,

(ii) $\dot{B} = H_{\psi_2} = A$

(iii) $\dot{\psi}_1 = -H_K = -\psi_1 \left(\dfrac{sf_K(K,L)}{c} + \left(\dfrac{B}{cK}\right)^2 r'\left(\dfrac{B}{cK}\right)\right)$

(iv) $\dot{\psi}_2 = -H_B = \dfrac{\psi_1}{c}\left(r\left(\dfrac{B}{cK}\right) + \dfrac{B}{cK}\, r'\left(\dfrac{B}{cK}\right)\right)$

(v) $H_{q_i} = e^{-\delta t}\, U_{q_i} - \psi_1 \dfrac{P_i}{c} = 0$

(vi) $H_L = \psi_1/c.(sf_L(K,L) - w) = 0$

(vii) $H_A = \psi_1/c + \psi_2 = 0$.

Now if as in Section 3 $f(K,L) = AK^{\alpha}L^{1-\alpha}$, $L = aK$ and $\frac{s}{c} f_K = b$,

so (vii), (iii) and (iv) give

$$r\left(\frac{B}{cK}\right) + \frac{B}{cK}\, r'\left(\frac{B}{cK}\right) = b + \left(\frac{B}{cK}\right)^2 r'\left(\frac{B}{cK}\right)$$

which can be solved for $Z^* = \left(\dfrac{B}{cK}\right)^*$, and so

$$g = r(Z) + Zr'(Z) = b + Z^2\, r'(Z) .$$

g has the interpretation of the total interest cost of an extra \$1 borrowed, i.e. the marginal rate of interest, and leads to the solution for $\psi_1(t)$ of

$$\psi_1 = \psi_o\, e^{-gt} .$$

Proceeding as in Section 3, the differential equation describing behaviour of the state variable K is

$$(4.4) \qquad \dot{K} = bK - \frac{P'\gamma}{c} - \frac{e^{(g-\delta)t}}{\psi_o} - r(Z)\ Z\ K + Z\ \dot{K}$$

$$= gK - \frac{P'\gamma}{(1-Z)c} - \frac{e^{(g-\delta)t}}{\psi_o(1-Z)}$$

with a general solution

$$(4.5) \qquad K(t) = \frac{e^{(g-\delta)t}}{\psi_o(1-Z)\delta} + \frac{P'\gamma}{gc(1-Z)} + C_3\ e^{gt}$$

C_3 a constant of integration.

Now $W(t) = cK - B$

$= cK - ZcK$

$= cK(1-Z)$

so the general solution for $W(t)$ is

$$(4.6) \quad W(t) = \frac{c\ e^{(g-\delta)t}}{\psi_o\delta} + \frac{P'\gamma}{g} + C_4\ e^{gt} \quad .$$

$W(t)$ is the state variable to which we should apply the transversality condition

$$\lim W(t) \geq 0 \quad .$$

Then $\lim \psi_3 (t) W(t) = 0$, where ψ_3 is the costate variable associated with W. From the economic interpretation of costate variables, it must be true that

$$\psi_3(t) = c\,\psi_1(t) = -\psi_2(t) \quad .$$

It is then clear that transversality gives $C_4 = 0$ i.e. $C_3 = 0$.

The state variable $W(t)$ can be used to synthesize controls. Then

$$(4.7) \qquad P_i q_i = P_i\gamma_i + \beta_i\ \delta(W - \frac{P'\gamma}{g})$$

$$= P_i\gamma_i + \beta_i \frac{\delta}{g}\ (gW - P'\gamma)$$

where gW may again have the interpretation of permanent income. In these terms we also have

$$(4.8) \qquad E = P'\gamma + \frac{\delta}{g}(gW - P'\gamma)$$

$$(4.9) \qquad L = aK = \frac{aW}{c(1-Z)}$$

$$(4.10) \qquad c\dot{K} = (g-\delta)cK + \frac{P'\gamma}{(1-Z)}\left(\frac{\delta}{g} - 1\right)$$

$$= \frac{(g-\delta)}{1-Z} W + \frac{P'\gamma}{1-Z}\left(\frac{\delta}{g} - 1\right)$$

$$(4.11) \qquad \dot{W} = (g-\delta)\left(W - \frac{P'\gamma}{g}\right) = \frac{g-\delta}{g}(gW - P'\gamma) \; .$$

These equations present an interesting choice. The variables Z and g are decision variables, and to be consistent with Section 3 we should parametrize their solutions in terms of the parameters of the production function and the average rate of interest function. In terms of investigating the effects of changes in prices (w, c, p, s, r) this reduction would be necessary. But Z and g may also be observables, of use in the empirical estimation of the functions.

5. A General Form of Production, Investment, Consumption and Financing Based on Multi-stage Maximization and the Theory of Duality

The results of the introductory models suggest that the set of necessary conditions in fact relate to a number of substantially separate optimization problems, each carried out at various "levels". Thus the model may be substantially generalized, and at the same time its structure simplified, by allowing for this multi-stage nature of the optimization process. Modern duality theory provides the proper framework within which this programme can be carried out. In effect the model can be decomposed into a number of modules, with the explicit structure of each module to some extent independent of other modules.

First let us look at the consumption sector, taking for the moment total expenditure, E, as given. Consider the following first-level maximization problem:

Given E and $P' = P_1, \ldots, P_n$, choose $q' = q_1, \ldots, q_n > 0$

to maximize

$$U(q)$$

subject to $P'q \leqslant E$. This maximization problem leads to the n demand functions:

$$q_i^* = q_i^* (E, P_1, \ldots, P_n)$$

and we can define the new function

$$V^*(E, P_1 \ldots, P_n) = U(q_1^*, \ldots, q_n^*)$$

$$= \max_q \{U(q) : P'q \leqslant E : q \geqslant 0\} .$$

V^* is the indirect utility function, and because of the linearity of the budget constraint V^* is homogeneous of degree zero in P and E, so it may be useful to introduce normalized prices

$$p_i = P_i/E$$

and the function

$$V(p_1, \ldots, p_n) = V^* (1, P_1/E, \ldots, P_n/E)$$

$$= \max_q \{U(q) : p'q \leqslant 1 : q \geqslant 0\} .$$

Duality theory relates the properties of U to those of V, and there is an even more complete duality between U and the reciprocal indirect utility function

$$H(p) = 1/V(p)$$

(see Diewert (1974) pp. 120-133). Thus we may either specify U and derive demand functions $q(p)$ by constrained maximization, or specify $V(p)$ and appeal to Roy's identity.

$$(5.1) \qquad q(p) = \frac{\nabla V(p)}{p'\nabla V(p)}$$

(see Diewert (1974) p.125). Diewert also gives three examples of possible functional forms for H(p) and the implied systems of demand equations. Econometrically, the possible endogeneity of E must be kept in mind, and it is E and $V^*(P,E)$ that relates this module to the other modules. But as far as intertemporal considerations are concerned, only $V^*(P,E)$ is relevant, since at each instant $U(q)$ is to be maximized subject to $P'q = E$. Thus in an intertemporal problem it is preferable to use $V^*(P,E)$ with E in the budget constraint, to avoid unnecessarily complicating the problem with conditions relating to maximization of $U(q)$.

On the production side, consider the following structure:

m capital goods

$$K' = K_1, \ldots, K_m$$

with corresponding purchase prices

$$c' = c_1, \ldots, c_m$$

ℓ variable inputs

$$L' = L_1, \ldots, L_\ell$$

with wage rates

$$w' = w_1, \ldots, w_\ell$$

and K outputs

$$Q' = Q_1, \ldots, Q_k$$

with selling prices

$$s' = s_1, \ldots, s_k \quad .$$

Production technology is given by the production possibility set $T = \{Q,K,L\}$, a point set of feasible input-output combinations. It is often useful to assume a frontier of the form $f(Q,K,L) = o$ which in the case of $k = 1$ is the usual production function. For our purposes, the distinction between capital goods and variable inputs is that capital goods are owned by the enterprise, requiring commitment of financial resources, whereas variable inputs are paid for out of current revenue. It will become clear that new capital goods compete directly with consumption, whereas variable inputs do not.

Duality theory suggests the following possible stacking of optimization problems:

(i) given K and Q, choose L to produce Q at minimum cost i.e.

$$C(K,Q,w) = \min_L \{w'L : (Q,K,L) \in T\}$$

C gives the minimum cost of producing various Q, given K, and the maximization procedure also leads to demand functions derived from C :

$$LC = LC(K,Q,w) \quad .$$

Duality theory relates C and f. Thus we may choose functional forms for C or f, and duality theory provides:

Shephard's Lemma:

$$LC_i(K,Q,w) = \frac{\partial C(K,Q,w)}{\partial w_i} \quad . \tag{5.2}$$

Diewert provides two examples of permissible cost functions. (Diewert (1974) pp. 109-120.)

(ii) Given K, choose Q, L to maximize net revenue, defined as $s'Q - w'L$ subject to production possibilities. This leads to the net revenue function.

$$NR(K,w,s) = \max_{Q,L} \{s'Q - w'L : (Q,K,L) \in T\}$$

and the maximization leads to demand and supply functions

$QNR\ (K,w,s)$

$LNR\ (K,w,s)$.

Note that

$$NR\ (K,w,s) = \max_{Q} \{s'Q - C\ (K,Q,w)\} \ .$$

There is a duality among NR (K,w,s), C(K,Q,w) and f(Q,K,L), so that any of these can be specified. Again there is an advantage in specifying NR, as duality theory gives: Hotelling's Lemma:

$$QNR_i = \frac{\partial NR}{\partial s_i}$$

$$(5.3) \qquad LNR_i = \frac{\partial NR}{\partial w_i} \ .$$

Also, LNR (K,w,s) = LC(K,QNR (K,w,s), w). N.R. is often referred to as the variable profit function, see Diewert (1974) pp. 133-141.

(iii) Given X, choose K to maximize NR (K,s,w) subject to the constraint $c'K = X$

i.e. $R(X,s,w,c) = \max_{K} \{NR(K,s,w) : c'K = X\}$

and we have the capital input demand functions

$KR(X,s,w,c)$

as well as variable input demand functions and output supply functions

$$LR(X,s,w,c) = LNR\ (KR(X,s,w,c),\ w,s)$$

$$QR(X,s,w,c) = QNR\ (KR(X,s,w,c),\ w,s) \ .$$

Although problem (iii) is not covered explicitly by the existing duality results, the existing results can be easily extended, and by analogy with $V^*(P,E)$,

$$(5.4) \qquad KR_i = \left(\frac{\partial R}{\partial c_i} \Big/ \Sigma c_i \frac{\partial R}{\partial c_i} \right) \cdot X$$

$$LR_i = \frac{\partial R}{\partial w_i}$$

$$QR_i = \frac{\partial R}{\partial s_i} .$$

(iv) Given the net worth of the agent, W, the agent can borrow against W to finance further expenditure on capital goods, i.e., given W, choose X and B subject to $W = X - B$. W is the true exogencus variable, and we wish to specify X and B as functions of W. Parts (i) through (iii) are of a purely static nature, but (iv) introduces an intertemporal problem.

(v) Given B,X, (iii) determines R (X,s,w,c), a source of funds. R is to be allocated among:

(a) Consumption expenditure E

(b) Investment in new capital goods $I = \dot{X}$

(c) Change in borrowing $\dot{B}$.

i.e. Choose E, I, $\dot{B}$ subject to $R + \dot{B} = E + I + r.B$

To introduce the intertemporal nature of the problem, all of the above variables are taken as functions of time. Since time is planning time, we initially take prices s,w,c to be constant, although the assumption of a constant rate of inflation should present no problems. Let the present be $t = o$, then in planning time $W(o)$ is given and the problem is to choose time paths for $E(t)$, $B(t)$, $X(t)$ to

$$\text{maximize} \int_o^\infty e^{-\delta t} \; V^* \; (E,P)$$

subject to

$$R(X(t)) + \dot{B}(t) = E(t) + \dot{X}(t) + r \,.\, B(t)$$

$$W(o) \text{ given}, \quad W(t) = X(t) - B(t)$$

$$\lim \; W(t) \geqslant o \,.$$

To proceed, functional forms for $V^*(E,P)$ and $R(X,s,w,c)$ must be specified, either directly, or indirectly through a utility function and a production function or a cost function or a net revenue function.

As an example of these ideas, we will work through problems (i) to (iv) for a Klein-Rubin utility function and a constant returns to scale Cobb-Douglas production function:

$$U(q) = \beta' \; \ell n \; (q-\gamma)$$

is to be maximized subject to $P'q \leqslant E$. Forming

$$\beta' \; \ell n \; (q-\gamma) + \lambda \; (E-P'q)$$

the necessary conditions are:

$$\frac{\beta_i}{q_i - \gamma_i} - \lambda \, P_i = o \qquad \text{i.e.} \quad P_i q_i = P_i \gamma_i + \frac{\beta_i}{\lambda}$$

and $P'q = E$.

Therefore $E = P'\gamma + \frac{1}{\lambda}$

giving $P_i q_i = P_i \gamma_i + \beta_i (E - P'\gamma)$

and $q_i = \gamma_i + \frac{\beta_i}{P_i}(E - P'\gamma)$

or $q_i = \gamma_i + \frac{\beta_i}{P_i} \quad (1 - p'\gamma)$.

(5.5) Now $V^*(P,E) = \Sigma \; \beta_i \; [\ln \beta_i - \ln P_i + \ln (E - P'\gamma)]$

and $V(p) = \Sigma \; \beta_i \; [\ln \beta_i - \ln p_i + \ln (1 - p'\gamma)]$.

Note that

$$q_i = \frac{\partial V}{\partial p_i} \Big/ p'\nabla V(p)$$

$$= \left(\frac{\partial V^*}{\partial P_i} \Big/ P'\nabla V^*(P,E)\right) . E \quad .$$

Now consider

$$Q = A\, K_1^{\alpha}\, K_2^{\beta}\, L^{\,1-\alpha-\beta} \; .$$

(i) Since only one L will produce Q for given K_1 and K_2, cost minimization is trivial and

$$L = \left(\frac{K_1^{-\alpha} \; K_2^{-\beta} \quad Q}{A}\right)^{\frac{1}{1-\alpha-\beta}}$$

with $C(K_1, K_2, Q, w) = wL^*$ so clearly labour demand based on the cost function satisfies

$$LC = \frac{\partial C}{\partial w} = w\frac{\partial L}{\partial w} + L, \quad \text{so} \quad \frac{\partial L}{\partial w} = o.$$

(ii) Given K_1, K_2 choose Q,L to maximize s Q - w L subject to the production function.

Form $s\,Q - w\,L - \lambda\,(Q - AK_1^{\alpha}\; K_2^{\beta} \; L^{1-\alpha-\beta})$

$s = \lambda$

$w = \lambda\; AK_1^{\alpha} \;\; K_2^{\beta} \;\; L^{-\alpha-\beta}\; (1-\alpha-\beta)$

$$\therefore \quad L = \left(\frac{sAK_1^{\alpha}\ K_2^{\beta}(1-\alpha-\beta)}{w} \right)^{\frac{1}{\alpha+\beta}}$$

$$Q = \left(\frac{s}{w}\right)^{\frac{1-\alpha-\beta}{\alpha+\beta}} A^{\frac{1}{\alpha+\beta}} (1-\alpha-\beta)^{\frac{1-\alpha+\beta}{\alpha+\beta}} K_1^{\frac{\alpha}{\alpha+\beta}} K_2^{\frac{\beta}{\alpha+\beta}}$$

are the demand and supply functions, say L^* and Q^*, and

$$NR\ (K_1, K_2, s, w) = sQ^* - wL^*$$

$$= sQ\ (K_1, K_2, s, w) - wL(K_1, K_2, s, w)$$

with, for example,

$$\frac{\partial NR}{\partial w} = s\ \frac{\partial Q}{\partial w} - w\frac{\partial L}{\partial w} - L\ .$$

Now since $s\dfrac{\partial Q}{\partial w} = s\dfrac{\partial Q}{\partial L}\ \dfrac{\partial L}{\partial w}$

$$= s(1-\alpha-\beta)\ AK_1^{\alpha}\ K_2^{\beta}\ L^{1-\alpha-\beta}\ \frac{\partial L}{\partial w}$$

$$= w\ \frac{\partial L}{\partial w}$$

by the first order condition, we have

$$\frac{\partial NR}{\partial w} = -L\ .$$

In this case,

$$NR(K_1, K_2, s, w) = A^{\frac{1}{\alpha+\beta}} \left(\frac{s}{w}\right)^{\frac{1}{\alpha+\beta}} w\ K_1^{\frac{\alpha}{\alpha+\beta}} K_2^{\frac{\beta}{\alpha+\beta}} (1-\alpha-\beta)^{\frac{1}{\alpha+\beta}} \frac{\alpha+\beta}{1-\alpha-\beta}$$

$$= b.\ \left(\frac{s}{w}\right)^{\frac{1}{\alpha+\beta}} w\ K_1^{\frac{\alpha}{\alpha+\beta}}\ K_2^{\frac{\beta}{\alpha+\beta}}\ , \quad \text{say.}$$

(iii) Given X, choose K_1, K_2 to maximize $NR\ (K_1, K_2, s, w)$ subject to $c_1K_1 + c_2K_2 = X$.

Let $NR = b\ \left(\frac{s}{w}\right)^{\frac{1}{\alpha+\beta}}\ w\ K_1^{\frac{\alpha}{\alpha+\beta}}\ K_2^{\frac{\beta}{\alpha+\beta}}$

and form

$$NR + \lambda\ (X - c_1 K_1 - c_2 K_2)$$

$$\lambda c_1 = \frac{\alpha}{\alpha+\beta}\ b\ \left(\frac{s}{w}\right)^{\frac{1}{\alpha+\beta}}\ w\ K_1^{\frac{-\beta}{\alpha+\beta}}\ K_2^{\frac{\beta}{\alpha+\beta}}$$

$$\lambda c_2 = \frac{\beta}{\alpha+\beta}\ b\ \left(\frac{s}{w}\right)^{\frac{1}{\alpha+\beta}}\ w\ K_1^{\frac{\alpha}{\alpha+\beta}}\ K_2^{\frac{-\alpha}{\alpha+\beta}}$$

and $\dfrac{c_1}{c_2} = \dfrac{\alpha}{\beta}\ \dfrac{K_2}{K_1}$

or $c_1 K_1 = \dfrac{\alpha}{\beta}\ c_2 K_2$

giving

$$c_1 K_1 = \frac{\alpha}{\alpha+\beta}\ X \quad , \quad c_2 K_2 = \frac{\beta}{\alpha+\beta}\ X \quad .$$

This gives the maximum revenue, for a fixed investment X in capital stock, as

$$R(X,s,w,c_1,c_2) = d\left(\frac{s}{w}\right)^{\frac{1}{\alpha+\beta}}\ w\ c_1^{\frac{-\alpha}{\alpha+\beta}}\ c_2^{\frac{-\beta}{\alpha+\beta}}\ X$$

$$\text{where} \quad d = A^{\frac{1}{\alpha+\beta}}\ (1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha+\beta}}\ \alpha^{\frac{\alpha}{\alpha+\beta}}\ \beta^{\frac{\beta}{\alpha+\beta}} \quad .$$

Note that X enters linearly because of overall constant returns to scale. To introduce non-constant returns to scale, the R function would be an appropriate starting place.

Sub problems (iv) and (v) could now be solved, using the two functions $R(X,s,w,c)$ and $V^*(P,E)$, much as in Section 4. Instead of doing this, we will now consider problems (iv) and (v) in a slightly more general model.

Let $t = o$ be the point of which planning time and actual time coincide, and at time zero the optimal plan for $o \leqslant t < \infty$ is to be specified. Given at time o is the initial m-vector of capital stocks, K_o, and the initial debt B_o. For simplicity assume static price expectations, so we write c for $c(o)$. Define the net worth state variable W by $W_o = c'K_o - B_o$. Note that while $K(t)$ and $B(t)$ may be discontinuous at $t=o$, we do have $W(o) = W_o$, since the transformation of one capital good into another, or the financing of more capital through borrowing, leaves W_o unaffected.

The multi-output, multi-input revenue function we use is a specialization of Diewert's translog variable profit function (Diewert (1974)):

$$(5.6) \qquad \ell n\ NR\ (s,w,K) = \alpha_o + \sum_{i=1}^{\ell} \alpha_i\ \ell n\ w_i + \sum_{j=1}^{k} \alpha_{\ell+j}\ \ell n\ s_j + \sum_{j=1}^{m} \eta_j\ \ell n\ K_j$$

with $\Sigma \eta_j = 1$, $\sum_{i=1}^{k+\ell} \alpha_i = 1$. This is dual to a multiple-output multiple-input production function, and the previous example is a special case of this form, with extra restrictions on the α, η. Now define the revenue function R by

$$R(s,w,c,X) = \max \quad NR(s,w,K) \text{ subject to } c'K = X .$$

Since ℓn is monotonic, the constrained maximization is simple and gives

$$c_j K_j = \eta_j X$$

and so

$$(5.7) \quad \ell n\ R = \alpha_o + \sum_{i=1}^{\ell} \alpha_i\ \ell n\ w_i + \sum_{j=1}^{k} \alpha_{\ell+j}\ \ell n\ s_j$$

... [continued overleaf]

[equation (5.7) concluded]

$$\ldots\ldots + \sum_{j=1}^{m} \eta_j \,\ell n\, \frac{\eta_j}{c_j} + \ell n\, X \ .$$

Note that X,K,s,w,c , and so also NR, R, can be functions of t in the above formulation. Efficient production given available resources is not an intertemporal problem, and so the product mix, input mix problem is relatively separate, being subsumed in the R function.

For the utility function we take a Klein-Rubin $U(q) = \beta' \,\ell n\, (q-\gamma)$. Then

$$V^{*}(P,E) = \text{constant} + \ell n\,(E-P'\gamma) - \Sigma\, \beta_i \,\ell n\, P_i \ .$$

There are now a number of state variables: K, B, $X = c'K$, $W = X - B$ and the derived state variable

$$Z = \frac{B}{X}$$

the debt-asset ratio of the decision making agent. Now let the average rate of interest per unit of debt be $r = r(Z)$, where r is specified to be a monotonic increasing function of Z . We also allow an external scurce of income y . Then the instantaneous flow budget constraint facing the consumer is

(5.8) $$y + R + \dot{B} = E + GI + r(Z).\, B + T$$

where GI is gross investment and T is taxes, defined by

$$T = u\,(R + y - u_I.GI - u_D\, D'_T\, \hat{c}\, K - u_r\, r(Z).B)\ .$$

u is the average rate of income tax, u_I the rate of investment tax credit, u_D the proportion of depreciation allowable for tax purposes, D_T a vector of allowable depreciation rates for tax purposes and u_r the proportion of interest allowable for tax purposes.

Now $GI = \dot{X} + D_A' c\hat{K}$

$= \dot{X} + D_A' \eta X$

since $c_j K_j = \eta_j X$ along an optimal path, where D_A is the vector of actual depreciation rates. The equation of motion for capital stock X can therefore be given by

$$\dot{X} = \frac{1}{1-uu_I} \Big[(y + R)\ (1-u) - E + \dot{B} - (1-uu_r)\ r\ (Z)\ B$$

$$- X \{D_A' \eta\ (1-uu_I) - uu_D\ D_T'\ \eta\} \Big] .$$

The two state variables are X and B, with a derived state variable Z, but in fact it will be true that $W = X-B$ will be the basic state variable. Controls are E and $\dot{B}$.

The problem can be posed as that of choosing time paths for the variables $X(t)$, $B(t)$, $E(t)$ and hence $R(t)$, $Z(t)$, $W(t)$ etc., to maximize

$$\int_0^\infty e^{-\delta t}\ V^*(P,E(t))dt$$

subject to the above constraint, the initial condition $W(o) = W_o$, and the terminal condition $\lim_{t \to \infty} W(t) \geqslant o$.

Introducing the dummy control $\dot{B} = A$, we form the Hamiltonian:

(5.9) $H(t,X,B;\ E,A:\ \psi_1,\ \psi_2)$

$$= e^{-\delta t}\ V^*(P,E) + \frac{\psi_1}{1-uu_I} [\ \{y + R\ (s,w,c,X)\}\ (1-u)$$

$$- E + A - (1-uu_r)\ r(Z)\ B - X\ \{D_A'\ \eta\ (1-uu_I) - uu_D D'_T \eta\}] + \psi_2\ A .$$

with necessary conditions:

(i) $\dot{X} = H_{\psi_1}$

(ii) $\dot{B} = H_{\psi_2} = A$

(iii) $\dot{\psi}_1 = - H_X = -[(1-u) \;.\; \text{constant}\; \Pi\, w_i^{\alpha} \, s_j^{\alpha_{\ell+j}} \, c_j^{-\eta_j} - \Theta$

$+ (1-uu_r)\, r'(Z)\, Z^2] \dfrac{\psi_1}{(1-uu_I)}$

(iv) $\dot{\psi}_2 = - H_B = \dfrac{\psi_1}{(1-uu_I)} (1-uu_r)\, (r(Z) + Zr'(Z))$

(v) $H_E = \dfrac{e^{-\delta t}}{E-P'\gamma} - \dfrac{\psi_1}{1-uu_I} = o$

(vi) $H_A = \dfrac{\psi_1}{1-uu_I} + \psi_2 = o$.

Conditions (vi), (iii) and (iv) together lead to a solution for Z, say Z^*, which is a function of all prices, tax parameters, the interest rate function, and the parameters of the production function, which can be assumed constant in planning time. Thus the two state variables, B and X, are optimally related by $B = Z^* X$. Now B and X are free end point state variables, but subject to the condition

$$X(o) - B(o) = W(o)$$

so the above optimal relation between B and X specifies them completely, i.e., $X - Z^* X = W$

$$X = \frac{W}{1-Z^*}$$

$$B = \frac{Z^*}{1-Z^*} X .$$

So once the optimal path of W is known, the optimal paths of B and X are determined. It is for this reason that W is the true state variable. Now the transition equation for W can be derived as follows:

$$\dot{W} = \dot{X} - \dot{B}$$

so

$$(1-uu_I)\ \dot{W} = (y+R)\ (1-u) - E + uu_I\ \dot{B} - (1-uu_r)\ r\ (Z)B - \Theta X$$

and

$$\frac{(1-uu_I - Z)}{1-Z}\ \dot{W} = y(1-u) - E + \frac{1}{1-Z}\ [(1-u)\ \text{constant} \prod_{i,j} w_i^{\alpha_i} s_j^{\alpha_j} c_j^{-n_j}$$

$$- \Theta - (1-uu_r)\ r(Z)\ Z\ ]\ W$$

where the optimal relations of B and X to W have been used, and the asterisk deleted from Z for simplicity.

Then

$$\dot{W} = \frac{1-Z}{1-uu_I-Z}\ .\ [y(1-u) - E + \frac{(1-uu_r)}{1-Z}\ \{(1-uu_I)\ (r(Z) + Zr'(Z)) - r'(Z)\ Z^2 - r(Z)Z\}W]$$

by the relation defining Z^*, and factoring gives

$$(5.10)\qquad \dot{W} = \frac{1-Z}{1-uu_I-Z}\ [y(1-u)-E] + (1-Z)\,(1-uu_r)\ (r(Z) + Zr'(Z)\ )W$$

$= a(1-u)y-aE + gW$, say. $\frac{g}{1-Z}$ has the interpretation of the marginal cost of borrowing \$1, equal to the marginal return of investing \$1, in new capital goods. Note that by condition (iv), condition (iii) can be written as

$\dot{\psi}_1 = -g\psi_1$ i.e. $\psi_1(t) = \psi_o e^{-gt}$, ψ_o constant.

Now from condition (v)

$$E = P'\gamma + \frac{(1-uu_I)}{\psi_1} e^{-\delta t} ,$$

i.e. ,

$$E = P'\gamma + \frac{(1-uu_I)}{\psi_o} e^{(g-\delta)t} \tag{5.11}$$

and so

$$\dot{W} = a(1-u)y - aP'\gamma - \frac{a(1-uu_I)}{\psi_o} e^{(g-\delta)t} + gW .$$

To proceed, the time path for $y(t)$, $o < t < \infty$, must be specified. We will assume y constant in planning time, although clearly other specifications, such as a constant exponential growth, could be considered. Then the general solution for $W(t)$ is

$$W(t) = \frac{e^{(g-\delta)t}}{\psi_o\delta} a(1-uu_I) + \frac{aP'\gamma-a(1-u)y}{g} + C_1e^{gt} .$$

Transversality gives $C_1 = o$, and the initial condition is

$$W(o) = \frac{a(1-uu_I)}{\psi_o\delta} + \frac{aP'\gamma}{g} - \frac{a(1-u)y}{g}$$

which then allows computation of the open loop path for $W(t)$:

$$W(t) = [W(o) - \frac{aP'\gamma}{g} + \frac{a(1-u)y}{g}] e^{(g-\delta)t} + \frac{aP'\gamma-a(1-u)y}{g} . \tag{5.12}$$

For econometric estimation, we need to synthesize the closed loop form of the controls. For total expenditure,

$$E(t) = P'\gamma + \frac{\delta}{a} [W(t) + \frac{a(1-u)y - aP'\gamma}{g}] . \tag{5.13}$$

Since $\dot{W}(t) = (g-\delta) [W(t) + \frac{a(1-u)y - aP'\gamma}{g}]$

and $B = \frac{Z}{1-Z} W$

$$(5.14) \qquad \dot{B}(t) = \frac{Z}{1-Z} \dot{W}(t) = \frac{Z}{1-Z}(g-\delta)\left[W(t) + \frac{a(1-u)y - aP'\gamma}{g}\right]$$

and

$$(5.15) \qquad GI(t) = \dot{X} + D_A' \eta X$$

$$= \frac{1}{1-Z}(g-\delta+D_A'\eta)\left[W(t) + \frac{a(1-u)y-aP'\gamma}{g}\right] .$$

Closed loop controls for E, $\dot{B}$ and GI are the major controls for the aggregates, determining the intemporal behaviour of the model. Duality theory allows the determination of subaggregates. For example, for categories of consumption

$$P_i q_i = P_i \gamma_i + \beta_i (E-P'\gamma)$$

$$= P_i \gamma_i + \beta_i \frac{\delta}{a}\left[W(t) + \frac{a(1-u)y-aP'\gamma}{g}\right]$$

and for variable inputs and outputs,

$$\frac{w_i L_i}{R} = \alpha_i$$

$$\frac{s_i Q_i}{R} = \alpha_{\ell+i}$$

while for fixed inputs

$$c_j \dot{K}_j = \eta_j \dot{X}$$

$$GI_j = \eta_j \dot{X} + D_{A_j} \eta_j X$$

$$= \frac{1}{1-Z}(g-\delta + D_{A_j})\eta_j \left[W(t) + \frac{a(1-u)y - aP'\gamma}{g}\right] .$$

Two comments are in order on closed loop controls (5.13) to (5.15) First, the role of y is slightly ambiguous. Strictly speaking, net worth W should incorporate y, but to do this from the beginning the problem of an appropriate capitalization rate arises. In the end we can interpret y either as added to net worth, so total net worth is $W(t) + \frac{a(1-u)y}{g}$, or as meeting part of the precommitted expenditure $P'\gamma$. Second, to define net worth out of which consumption takes place it may seem more appropriate to use a measure of "maximized net worth" based on an optimal policy rather than a market value concept, $c'K - B$. However, the point is that, if there were a perfect financial capital market, the two would be linearly related, and in the absence of a perfect capital market the former measure is meaningless. Thus $W(t)$ as used is appropriate.

Three variables which appear are a,g and Z, where

$$g = (r(Z) + Zr'(Z))\ (1-uu_r)\quad (1-Z)$$

and $$a = \frac{1-Z}{1-uu_I-Z}\ .$$

When Z appears on the R.H.S., it is really Z^*, the optimal debt-capital ratio. Z being optimal implies the cost of borrowing an extra dollar, determined by $r(Z)$, is equated to the marginal return of a dollar invested in capital goods, which depends on the production function, prices, tax laws. Thus we do not yet have a reduced form model, and the endogeneity of Z (and hence of a and g) must be faced. This will be considered in Section 7.

6. Behaviour in Calendar Time

Equations (5.13) (5.14) (5.15) represent the closed loop control formulation for the dynamic component of the model, giving controls at each point in time as a function of the state at that time, the relevant state being W. An initial decision is made to bring X and B into an optimal relationship. From then on, provided prices, the production function, interest schedules etc., do not change, following equations (5.13), (5.14) and (5.15) will map out the optimal path.

Closed loop controls also have the property that should any of these exogenous variables change, optimal controls are still generated, since the relevant prices etc. appear on the r.h.s. Thus, for example, equation (5.13) always gives optimal expenditure.

More care is required with (5.14) and (5.15). Consider, for example, a change in the interest schedule $r(Z)$. Then Z^* will change and $\dot{B}$ from (5.14) will generate the optimal rate of change for B to stay on an optimal path. But changing Z^* requires a change in the relation between B and X, i.e. a discrete movement to a new path as well. The equation

$$B = \frac{Z}{1-Z} W$$

determines this change. Because of this implication, the variables X and B, previously identified as state variables, become in a way control variables. This is because they are free-endpoint state variables, and are capable of adjustment to some extent like control variables. Once this adjustment in B has taken place, equation (5.14) generates optimal movement along the new path.

A change in c_j has an even greater impact, because it affects W as well as Z and so the change to a new path has this added effect.

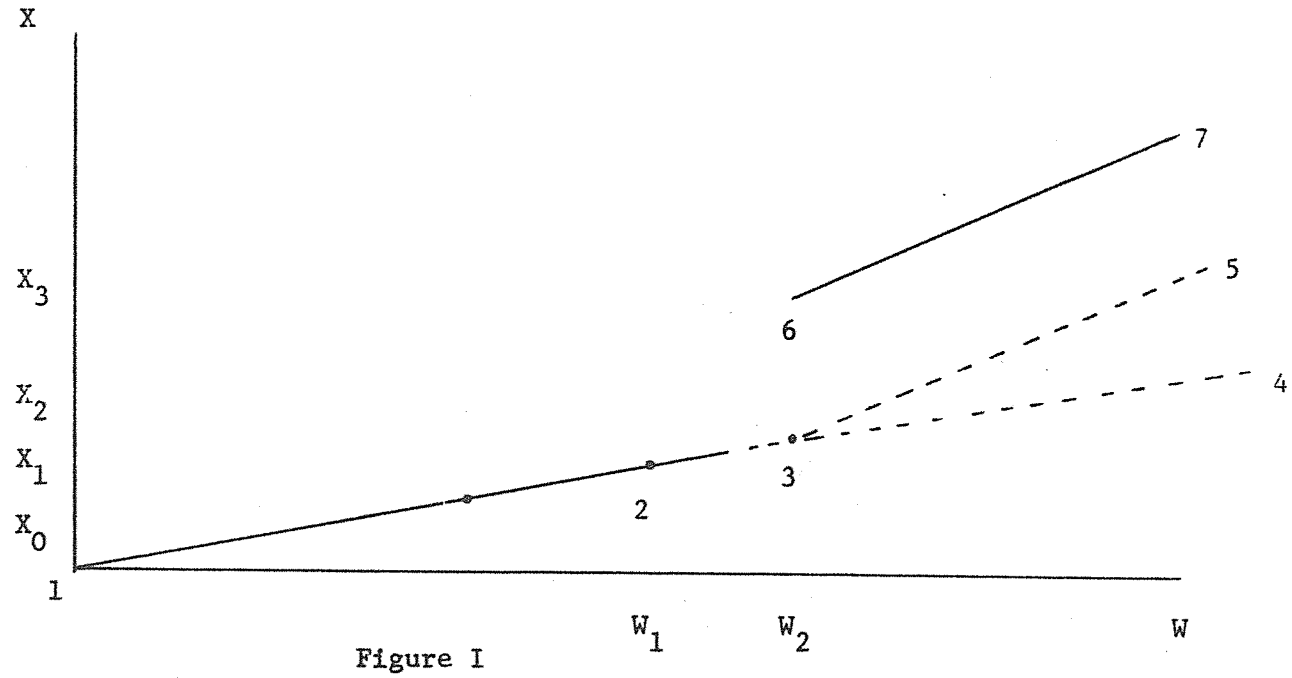

Figure I

For given prices, let $Z^* = Z_1$ and the optimal relation is $X = \frac{W}{1-Z_1}$

Prices stay constant until t_1, at which time $W = W_1$. Without a change in prices, path 1234 would be followed. At time t_1, c changes and W jumps to W_2, with Z changing to Z_2. If at this point rule (5.15) is applied, path 35 would be followed, which is not an optimal path, since (5.15) only preserves an optimal path once it is achieved. To move to a new optimal path requires the reallignment of X and B, which is represented by the jump from 3 to 6. Once at 6, (5.15) can be followed, and will generate the optimal path 67.

This suggests that it may be better to treat X, B and E as controls rather than $\dot{B}$ and E to derive the synthesized result, since such controls will always give the optimal relation. That is,

$$X = \frac{W}{1-Z}$$

To derive the implications for net investment:

$$\frac{dX}{dt} = \frac{\partial X}{\partial W}\frac{dW}{dt} + \frac{\partial X}{\partial Z} \cdot \frac{dZ}{dt} .$$

Now $W = X - B = c'K - B$

$$\frac{dW}{dt} = \dot{c}'K + c'\dot{K} - \dot{B}$$

$$= \dot{c}'K + \dot{W}$$

(where a $\cdot$ has been used for time derivative along an optimal path).

so $\frac{dX}{dt} = \frac{1}{1-Z}\dot{W} + \frac{1}{1-Z}\dot{c}'K + \frac{W}{(1-Z)^2}\left(\frac{\partial Z}{\partial c}\dot{c} + \frac{\partial Z}{\partial s}\dot{s} + \frac{\partial Z}{\partial w}\dot{w} + \right.$ effects due to changes in borrowing rules, tax rules etc)

which decomposes net investment into its three components: movement along an optimal path, movement to a new optimal path due to capital gains, and refinancing due to the gain in leverage. This interpretation depends partly on the timing of price changes, but one example of these three parts is the movement $X_o \to X_1 \to X_2 \to X_3$ on Figure 1 . Of course the second of these components does not enter into the investment function, being an automatic revaluation of capital stock. This term is missing from $\frac{dB}{dt}$

$$\frac{dB}{dt} = \frac{Z}{1-Z}\dot{W} + \frac{W}{(1-Z)^2}\left(\frac{dZ}{\partial c}\dot{c} + \text{ etc}\right) .$$

This analysis applies to a change in c . E will exhibit a behaviour more like $\frac{dX}{dt}$ if durable goods are included. (This generalization of E.L.E.S. has been carried out by Dixon and Lluch (1975) and can easily be integrated into this model through the appropriate module.)

At this point a problem must be faced. For movement along a continuous path in planning time, the above model is correct. But in moving <u>onto</u> a new planned path, due to a change in prices etc. the constraint $W_o = X_o - B_o$, $W(o) = W_o$ $W(o) = X(o) - B(o)$ is no longer accurate, because extra borrowing to finance new capital stock will attract an investment tax credit i.e., the appropriate constraint is

$$(X(o) - X_o) \quad (1-uu_I) = B(o) - B_o \quad .$$

Comparing this condition with necessary condition (vi) shows that with this constraint, the transversality condition is satisfied. Similar problems arise with a possible reallocation of X among K_j.

Thus under the interpretation used so far, $W(o) \neq W_o$. The easiest way to overcome this problem is to redefine c_j as $c_j\ (1-uu_I)$ if good j qualifies for tax credit, so that a change in u_I changes these c_j . Then $W = c'K$ is continuous in planning time for any K_o , B_o and all we need do is set $u_I = o$ and reinterpret c_j in previous results. Of course the one way effect of u_I is still a problem, so our results strictly hold only for a continually expanding enterprise. The parameter a is now set at 1.

7. Some Suggestions for Empirical Implementation

Let the vector D_A be given, and assume observations on categories of activity at certain points in time. Within each category, the net revenue function (5.7) can be estimated by regressing $\ln R - \ln X$ on $\ln w_i$, $\ln s_i$, $- \ln c_j$ while alternative estimates can be obtained from

$$\frac{w_i L_i}{R} , \quad \frac{s_i Q_i}{R} , \quad \frac{c_j K_j}{X} \quad .$$

At each point in time, presumably each category faces the same $r(Z)$ function. Approximate $r(Z)$ by $r(Z) = r_1 + r_2 Z$ and estimate r_1 and r_2 for each time point. r_1 and r_2 are assumed truly exogenous.

Now the optimal Z, for this form of $r(Z)$, is given by the equation:

$$(1-u) \text{ constant } \Pi \; w_i^{\alpha} \, s_j^{\alpha_{\ell+j}} \; c_j^{-\eta_j} - \Theta + (1-uu_r) \, r_2 Z^2 = (1-uu_r) \; (r_1 + 2r_2 Z)$$

which gives two possible solutions for Z :

$$(7.1) \qquad Z_{1,2} = 1 \pm 2 \sqrt{r_2^2 \, (1-uu_r)^2 - (1-uu_r) \, r_2 \, [(1-u) \text{ con}\Pi w^{\alpha} s^{\alpha} c^{\eta} - \Theta - (1-uu_r) r_1]}$$

and since Z should be real and $0 < Z < 1$, the negative root is taken. Note that the first term in square brackets is $(1-u)$ by the "purged" R/X series. Thus while the introduction of an imperfect capital market creates substantial non-linearities in reduced form equations, this non-linearity is definitional rather than an estimated equation. This generates a purged Z series for each category of activity. On the other hand, over the sampling period, actual observations on Z are available from X, B, W, so that (7.1) could be used as an estimating equation, except that everything on the R.H.S. of (7.1) is already given. Since any use of (7.1) to estimate parameters of $r(Z)$ or R would be complicated, this over-identifying restriction may have to be ignored. Thus there are two possible ways of generating a purged Z series: by (7.1) for each category of activity through time; or at each point in time and for each activity averaging B/X across observations. For estimation, the second of these may be preferable, being less affected by mis-specifications of $r(Z)$ and $R(s,w,c,X)$. But for policy predicitions, only (7.1) is available.

Given the purged Z series for each category, a series on g can be computed and we are at a stage to estimate the parameters determining the inter-temporal nature of the problem. Actually, the only parameter of an inter-temporal nature is δ, which occurs in equations (5.13), (5.14), (5.15) determining E, $\dot{B}$ and GI. If we were to estimate these equations, the considerations of Section 6 as to the calendar time "jumps" in $\dot{B}$, GI would be relevant. But it was shown there that exactly the same behaviour can be generated by considering

$$B = \frac{Z}{1-Z} W$$

$$X = \frac{1}{1-Z} W$$

which, with the purged Z series, have no empirical content. Thus only (5.13) is relevant to estimate δ. It appears that δ appears in (5.14) and (5.15) precisely because of the adding up property imposed by the budget constraint, so δ appears there only through its appearance in (5.13). For policy considerations, the jump terms will be important, but we can either take the approach of Section 6, or use, for example,

$$B = \frac{Z}{1-Z} W$$

so $\Delta B = \left(\Delta \left(\frac{Z}{1-Z}\right).\right) W + \frac{Z}{1-Z} \Delta W$

and all the jump terms are in $\Delta \left(\frac{Z}{1-Z}\right)$.

Now all that remains is to estimate (5.13). But this can be done precisely by the methods used to estimate E.L.E.S. Thus the whole model can be estimated by existing techniques.

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