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AN APPROACH TO THE MACROECONOMIC CLOSURE OF GENERAL EQUILIBRIUM MODELS

by

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INTRODUCTION

Quantitative economic models designed for use in policy analysis can be broadly classified into two distinct categories. First, there are the large scale applied general equilibrium models, which explain the composition of economic activity at the level of disaggregation provided by input-output tables. Conditional on a given macroeconomic climate, such models provide sectoral specific responses to shocks imposed on exogenous variables (which may be of an aggregative or a sectoral nature). Second, there are the small to medium scale dynamic macroeconometric models which attempt to explain the main macro-aggregates at the national accounts level of disaggregation. Models in the latter class can be used to compute the responses of various aggregates, such as total employment and total output, to given shocks. While each of these models is useful in its own right, and many policy applications of each could be quoted, an important potential which seems not to have been explored in the literature is the possibility of using an existing macroeconomic model to provide an endogenous macroeconomic climate for a general equilibrium model. Consider, for example, a shock such as a 10% increase in government spending. The general equilibrium model will in general provide the sectoral specific responses to the "demand" component of this shock, conditional on the remaining macroeconomic variables being held fixed. The macroeconomic model should be capable of modelling, for example, the financial aspects of the increased government spending. In general, the macro model may be used to close the general equilibrium model and can hence pass the effects on such variables as consumption, investment and the exchange rate on to the latter model.

Two major problems arise in the use of a macro model to close a general equilibrium model. One is temporal aggregation. Because macro models are typically dynamic, whereas general equilibrium models are not, care must be taken with the timing of the macroeconomic responses which are fed in to the general equilibrium model. The second is computability. Because realistic general equilibrium models are typically quite "large", with a developed methodology for solution, it is important that the proposed methodology for macroeconomic closure does not modify the model in such a way as to make it unusable for policy analysis. Ideally, the closed model should be used in much the same way as the existing model is currently used. In this paper we propose solutions to these problems in the case in which the macro model is of the Bergstrom class (see Bergstrom (1976)) and the general equilibrium model is of the Johansen class (see Johansen (1960)), but of course the general approach could be applied to other classes of models. The theory developed is then illustrated by the use of two models of the Australian economy. The general problem arose as part of the IMPACT Project,[1] and its solution led to the results reported in this paper.

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THE TWO MODEL TYPES

The Bergstrom Model

This is a macroeconomic model specified as a first order stochastic differential equation system. It may be represented as:

$$\dot{Y}_B(t) = A_B Y_B(t) + N_B Z_B(t) + u(t) , \qquad (1)$$

where Y_B refers to a vector of the logarithms of endogenous variables, Z_B is a vector of the logarithms of exogenous variables and u is a white noise disturbance vector. The typical equation specification is one of partial adjustment of endogenous variables to their desired or long-run values. An example of a model specified in the form of equation (1) is provided by Bergstrom and Wymer (1976).

A more rigorous specification of the underlying stochastic differential equation system is :

$$d\, Y_B(t) = A_B Y_B(t)dt + N_B Z_B(t)dt + dv(t) , \qquad (2)$$

where $dv(t)$ is a Gaussian vector process. The solution of this system may be written:

$$Y_B(t) = e^{A_B t} \{Y_B(0) + \int_0^t e^{-A_B \tau} N_B Z_B(\tau)d\tau + \int_0^t e^{-A_B \tau} dv(\tau)\} . \qquad (3)$$

Let $Z_B(\tau) = Z_B^c(\tau)$ a cet. par. value of a time path of exogenous variables. The control values of Y_B from (3) are:

$$Y_B^c(t) = e^{A_B t} \{Y_B(0) + \int_0^t e^{-A_B\tau} N_B Z_B^c(\tau) d\tau + \int_0^t e^{-A_B\tau} dv(\tau)\} . \tag{4}$$

Let $Z_B^s(\tau) = Z_B^c(\tau) + z_B$, with z_B a constant vector of exogenous shocks. Then the shocked values of Y_B are:

$$Y_B^s(t) = e^{A_B t} \{Y_B(0) + \int_0^t e^{-A_B\tau} N_B Z_B^s(\tau)\, d\tau + \int_0^t e^{-A_B\tau} dv(\tau)\}. \tag{5}$$

Defining the deviation of the shocked value from the control value by lower case letters, (5) - (4) gives:

$$y_B(t) = [\int_0^t e^{A_B(t-\tau)} d\tau]\, N_B\, z_B . \tag{6}$$

Defining:

$$C_B(t) = [\int_0^t e^{A_B(t-\tau)} d\tau]\, N_B = A_B^{-1} [e^{A_B t} - I] N_B, \tag{7}$$

we may write (6) as:

$$y_B(t) = C_B(t)\, z_B. \tag{8}$$

The interpretation of (8) is as follows. Since Y_B and Z_B are logarithms of variables, y_B and z_B are proportional changes. The latter, z_B, represent proportional changes which occur in the levels of the exogenous variables Z_B at time 0 and which are maintained throughout the

4 For reasons of space we do not here attempt a rationalization of the differences between the two columns of Table 1. Given an acceptance of the more inflationary effect of the increase in government spending when ORANI is closed by MACRO, however, the two other major differences are immediately explicable: in ORANI, trade flows are quite sensitive to changes in the domestic price level, although the story cannot be told properly without also making reference to the exchange rate changes endogenized by MACRO.

FOOTNOTES

* The authors wish particularly to thank Alan Powell, who first posed the problem to us, for his encouragement, constructive criticism and guidance. Thanks are also due to Peter Dixon, Brian Parmenter, Russell Rimmer and Dennis Sams at IMPACT, and to Peter Jonson, Rob Trevor, Warwick McKibbin and John Taylor of the Reserve Bank of Australia for their helpful contributions; Vivian Lees provided computational assistance. Of course, the authors accept full responsibility for any possible misuse or misunderstanding of either the ORANI or RBII models.

1 For background material on the aims and strategy of the IMPACT Project, see Powell (1977). For a statment of the arguments on the choice of the types of model used in IMPACT, see Powell (1981).

2 It is not strictly necessary to restrict β to values less than 1. Since (16) is only meant to provide a dynamics for the Johansen model within the strictly finite solution period t^*, values of β exceeding one do not cause the model to become unstable.

3 A possible methodology for experimental joint determination of (t^*, β) is outlined in Cooper and McLaren (1981a); an approach involving their direct econometric estimation is sketched in Cooper and McLaren (1981b).

interval $[0,t]$. The former, $y_B(t)$, represent the accumulated changes in the Bergstrom model's endogenous variables up to the point in time t expressed as a proportion of the <u>ceteris paribus</u> values of these variables at time t. Thus we note that one implication for policy analysis of the representation (8) is the ease with which a response to a sustained shock can be computed. By using (7), responses (8) can be calculated explicitly for any given value of t, and hence an approximate discrete time dynamic simulation of (1) is not required.

The Johansen Model

In its reduced form, a general equilibrium model may be characterised by the set of equilibrium conditions relating the levels of the endogenous variables Y to the levels of the exogenous variables Z

$$Y = f(Z). \tag{9}$$

Movement from one equilibrium position to another, generated by a small change in the exogenous variables Z, is represented as :

$$dY = f_{Z'}(Z)dZ \quad , \tag{10}$$

where the ij^{th} element of the Jacobian is $\partial f_i(Z)/\partial Z_j$, f_i is the i^{th} element of the vector function f, and Z_j is the j^{th} element of Z. A Johansen model is characterized by the assumption that the reduced form elasticities are constant for practical purposes. Thus (10) is written in proportional change form as:

$$y_J = C_J z_J , \quad (11)$$

where C_J is treated as a constant matrix determined by the behavioural parameters of f and the conditions prevailing in the initial equilibrium.

Although there is no explicit analysis of the path by which y_J is achieved in a Johansen model, some assumption about the length of time (to be referred to as t^*) needed for y_J to eventuate, is implicit in the estimation of the elasticities C_J and may also be related to the choice of the endogenous/exogenous split.

Closure of the Conditional General Equilibrium Model

A comparison of (8) and (11) reveals the potential for the use of a Bergstrom model to effect the closure of a Johansen model which has been constructed as a general equilibrium model conditional on an exogenous environment which can be endogenized by the Bergstrom model: the final form of each model is expressed in a form which is linear in percentage changes.

CONCLUSIONS

The potential policy applications of general equilibrium models would be considerably enhanced if these models could be linked, at the simulation stage, with existing macroeconometric models in such a way as to endogenize the macroeconomic climate. In such a closure, a major problem which arises is that of timing. With a dynamic macro model, responses accumulate continually throughout the response interval of the general equilibrium model, and hence cannot be treated as "time zero" shocks. The solution is to endow the general equilibrium model with a dynamic structure. Because data problems limit the estimation of explicit dynamics, a simple parameterisation is required in order to solve the timing problem, at least to a first order of magnitude. In the case of linking a Johansen model to a Bergstrom model, it has been shown that one particular parameterisation has many advantages, the major one being that it reduces the computational problems to an order equivalent to what is generally involved in the use of a Johansen model in "stand-alone" form. The procedure was illustrated with an application to two models of the Australian economy.

Table 1

Comparative ORANI Responses[(a)]

ORANI Endogenous Variables	ORANI Response (%)	
	Stand Alone	Closed by MACRO
Output	1.060	0.963
Prices	1.259	5.213
Labour Demand	1.555	1.389
Value of Exports	-1.364	-2.273
Value of Imports	1.832	3.254

(a) Responses are to a sustained shock to real government spending of 10%. The calculated "as if" shocks (in the variables exogenous to ORANI which are endogenized by MACRO) are – exchange rate: 1.65% devaluation of the Australian dollar ; real wages : up 0.43% ; consumption demand: up 0.71% ; investment demand: up 0.20% .

NOTATION FOR THE CLOSURE

To develop a notation which is consistent in its treatment of the endogenous/exogenous variables in the Johansen/Bergstrom models, consider first the vector y_B of all variables endogenous to the full Bergstrom system. The variables in this set may also be classified according to whether they are:

i) endogenous to the Johansen model. Denote this set of variables by $y_B y_J$.

ii) exogenous to the Johansen model. Denote this set of variables by $y_B z_J$.

iii) not included in the Johansen model. Denote this set of variables by $y_B \tilde{x}_J \equiv y_B - y_B x_J$, where x_J refers to the set of all variables in the Johansen system.

Thus the vectors y_B, y_J, z_B and z_J can be decomposed as:

$$y_B = \begin{pmatrix} y_B y_J \\ y_B z_J \\ y_B \tilde{x}_J \end{pmatrix}, \quad y_J = \begin{pmatrix} y_J y_B \\ y_J z_B \\ y_J \tilde{x}_B \end{pmatrix}, \quad z_B = \begin{pmatrix} z_B y_J \\ z_B z_J \\ z_B \tilde{x}_J \end{pmatrix}, \quad z_J = \begin{pmatrix} z_J y_B \\ z_J z_B \\ z_J \tilde{x}_B \end{pmatrix}.$$

In this notation, the vector $y_B y_J$ contains the same set of elements as $y_J y_B$, etc. Thus the first block of elements of y_B is the same as the first block in y_J, the second in y_J the same as the first in z_B, the second in z_B the same as the second in z_J, and the second in y_B the same as the first in z_J. The vectors y_B and y_J respectively contain all the variables endogenous to the Bergstrom model and to the Johansen model. The three sub-categories of endogenous variable represented by the partitioning are: endogenous to both models, endogenous to own model but exogenous to other model, and endogenous to own model but not appearing in other model. The three subcategories of the exogenous variable sets z_B and z_J are:

exogenous to own model, but endogenous to other model, exogenous in both models, and exogenous to own model but not appearing in other model.

While the vectors y_By_J and y_Jy_B contain the same <u>set</u> of variables, the ordering will be used to distinguish the numerical values of these variables. Thus y_By_J identifies the vector of <u>values</u> of the doubly endogenous variables as endogenized by the Bergstrom model, while y_Jy_B refers to the values of these variables as endogenized by the Johansen model. Similarly, while z_Jy_B and y_Bz_J refer to the same set of variables, z_Jy_B refers to a shock to certain variables exogenous to the Johansen model, whereas y_Bz_J gives the response of certain variables endogenous to the Bergstrom model as a result of a shock to variables in the set z_B. Of course the aim of constructing a combined model is to identify y_Bz_J with z_Jy_B in the closed system.

To this stage the notation has not been related specifically to the case where one model is being used to close the other. However, since we are interested in analysing such a special case in which the Bergstrom model closes the Johansen model, the notation for the conceptual combination of the models may be simplified by noting that in this case the set y_Jz_B is null.

<u>Results</u>

For the results presented in this section, the shock consists of a 10% increase in government spending. Thus z_0 consists of a vector of all zeros except for an entry of 0.1 in the appropriate position. The ORANI "stand-alone" response can be calculated simply as C_0z_0, and a sample of these results is presented in Table 1. Government spending is also exogenous to MACRO, and hence is an element of z_M. The shock to government spending will produce a time varying MACRO response, $C_M(t)z_M$, and this vector will contain variables such as the exchange rate, real wages, consumption and investment which are exogenous to ORANI and were hence implicitly set equal to zero in the calculation of the ORANI "stand-alone" responses. To account for these secondary effects of the shock to government spending, equation (33) details the construction of the "as if" shocks $\overline{z_0y_M}$, which then allows the calculation of the ORANI response to the endogenized macroeconomic climate to be calculated simply by multiplying the elements of the "as if" shock $\overline{z_0y_M}$ by the appropriate elements of the ORANI elasticities matrix C_0. These responses are added to the ORANI "stand-alone" responses, to give the "ORANI closed by MACRO" responses which are also given in Table 1. A comparison of the two sets of results demonstrates the potential importance of the possibility of constructing the macroeconomic closure of a general equilibrium model.[4]

variables (the "macroeconomic climate") which are exogenous to ORANI are endogenized in the joint model by MACRO: the exchange rate, real wages, private consumption demand, private investment demand.

In order to use MACRO to close ORANI, it remains to specify t^*, the ORANI stand-alone conditional equilibrium response time interval, and the value of β, the ORANI adjustment speed. For this illustrative example, we set $t^* = 8$ quarters and $\beta = 0.5$, which are certainly reasonable values. In the following pages, where earlier definitions are used, the J subscript will be replaced by an 0 (for ORANI) and the B subscript will be replaced by an M (for MACRO).

THE MODELS COMBINED

Rewriting (11) and (8) in the new notation gives:

$$\begin{matrix}(a)\\(b)\end{matrix}\begin{pmatrix} y_J y_B \\ y_J \tilde{x}_B \end{pmatrix} = \begin{bmatrix} C_J^{11} & C_J^{12} & C_J^{13} \\ C_J^{21} & C_J^{22} & C_J^{23} \end{bmatrix} \begin{pmatrix} z_J y_B \\ z_J z_B \\ z_J \tilde{x}_B \end{pmatrix} \qquad (12)$$

$$\begin{matrix}(a)\\(b)\\(c)\end{matrix}\begin{pmatrix} y_B y_J \\ y_B z_J \\ y_B \tilde{x}_J \end{pmatrix} = \begin{bmatrix} C_B^{11} & C_B^{12} \\ C_B^{21} & C_B^{22} \\ C_B^{31} & C_B^{32} \end{bmatrix} \begin{pmatrix} z_B z_J \\ z_B \tilde{x}_J \end{pmatrix} . \qquad (13)$$

In principle the system (12) may be closed with respect to $z_J y_B$ by identifying these variables with the set $y_B z_J$ endogenized by (13) (b). However, it is convenient to consider firstly a simplified version of the recursive structure in which $z_J = y_B$. In this case the models (12) - (13) simplify to:

$$\begin{matrix}(a)\\(c)\end{matrix}\begin{pmatrix} y_J y_B \\ y_J \tilde{x}_B \end{pmatrix} = \begin{bmatrix} C_J^{11} \\ C_J^{21} \end{bmatrix} z_J y_B \quad , \qquad (12^*)$$

$$(b) \quad y_B z_J = \begin{bmatrix} C_B^{21} & C_B^{22} \end{bmatrix} \begin{pmatrix} z_B z_J \\ z_B \tilde{x}_J \end{pmatrix} \quad , \qquad (13^*)$$

in which the fully recursive structure is revealed by the identity of the left-hand variable of (13*) with the right-hand variable of (12*).

TIMING ASPECTS OF THE CLOSURE

The Dynamics Behind the Johansen Model

Recognizing the fundamental similarity of (8) to (11), it is convenient to define matrices A_J and N_J such that:

$$C_J = A_J^{-1} [e^{A_J t^*} - I] N_J \ , \qquad (14)$$

where t^* is the Johansen response interval. Clearly, without further restrictions A_J and N_J are not unique.

Given (14), a natural assumption concerning the dynamics of the Johansen response is:

$$C_J(s) = A_J^{-1} [e^{A_J s} - I] N_J, \quad 0 \leq s \leq t^* \ . \qquad (15)$$

Thus:

$$y_J(s) = [e^{A_J s} - I][e^{A_J t^*} - I]^{-1} \ y_J(t^*), \ 0 \leq s \leq t^* \ , \qquad (16)$$

illustrates the fact that equilibrium is approached along a multivariate analogue of a path along which the (time) rates of change of the Johansen proportional-change responses decay geometrically (see p. 17 below).

Once the Johansen model has an assumed dynamic structure, linking of (13) to (12) or equivalently of (13^*) to (12^*) is feasible. The rest of this

AN ILLUSTRATIVE EXAMPLE

Model Specification

The theory of the preceding sections may be illustrated by the use of two models of the Australian economy. The Johansen model is to be identified with ORANI, which is fully documented in Dixon *et al.* (1982). ORANI is a general equilibrium model, of the Johansen family, specifying the sectoral composition of output, the aggregate volumes and composition of imports and exports, occupation specific demands for labour, and relative commodity prices. The model specifies 113 industries, 115 commodities and 9 skill categories of labour. Hence the elasticities matrix is quite "large". Although the allocation of variables to the endogenous/exogenous categories is user-determined, in typical experiments the macro-economic climate is taken as given, and the following macro variables are typically included in the exogenous list: real wages, exchange rate, private consumption demand, private investment demand, government spending.

The Bergstrom model is to be identified with one of the Reserve Bank of Australia's RBII class of models, and the specific version used is documented in Jonson and Trevor (1981). This version of the model will be designated as MACRO. MACRO is a disequilibrium continuous time model of the Bergstrom type, with 26 endogenous variables and 47 exogenous variables. It models such macro-aggregates as consumption, investment, demand for and supply of labour, output, exports, imports, wages, prices, taxes, the exchange rate, various interest rates and various monetary aggregates. MACRO has been estimated from time series data by the use of full information maximum likelihood methods.

Both ORANI and MACRO were extended by the addition of identities to introduce compatibility of variable definition (for details, see Cooper and McLaren (1980)). With these specifications of the models, the following induced by the linkage to the Bergstrom model can be computed simply in two stages. At the first stage, the "as if" shock $\overline{z_J y_B}[0,t^*]$ is calculated. Note that this involves only the parameters of the Bergstrom model, contained in A_B and N_B, and the two parameters β and t^*. The typically "large" matrix C_J is not involved. At the second stage, the "as if" shock is treated in exactly the same way as it would be in computing a Johansen "stand-alone" response.

To make this procedure operational, the two parameters t^* and β need to be specified. Values of these parameters may to some extent be implicit in the construction of the Johansen elasticities matrix C_J, and so the model-builder may be able to provide some prior evidence as to their values.[3]

section explains the way in which timing differences between the two models may be corrected, by decomposing a time path of shocks into a sequence of constant sustained shocks.

The Recursive Linkage: A Simplified Case

We consider firstly the case $z_J = y_B$. Thus the combined model, uncorrected for timing, is (12*) - (13*). In this sub-section the shorthand notations y_J, y_B, z_J, z_B, C_J, C_B are understood to represent the vectors and matrices of the system (12*) - (13*).

Consider now a shock to the Bergstrom model which occurs at time 0 and is sustained over $[0,t]$, $t < t^*$. Represent this by $z_B[0,t]$. This produces the Bergstrom response:

$$y_B(s) = C_B(s) z_B[0,t], \quad 0 < s < t, \tag{17}$$

where

$$C_B(s) = A_B^{-1} [e^{A_B s} - I] N_B.$$

Identifying y_B with z_J leads to a time varying shock $z_J(s)$ to the Johansen model. Representing $z_J(s)$ as the integral of its derivatives:

$$z_J(s) = \int_0^s \left(\frac{dz_J(\tau)}{d\tau}\right) d\tau \tag{18}$$

allows each of the derivatives to be interpreted as a shock sustained over the period $[\tau,t]$:

$$\left.\frac{dz_J(s)}{ds}\right|_{s=\tau} \equiv z_J[\tau,t], \quad 0 < \tau < t. \tag{19}$$

The effect of this latter shock on the Johansen model can be derived by use of (16), with $s = t - \tau$. This effect is:

$$A_J^{-1} [e^{A_J(t-\tau)} - I] N_J z_J[\tau,t] . \qquad (20)$$

Because of the linearity of the Johansen model, its responses to any given shock may be obtained as the sum of the individual responses to a series of smaller shocks which cumulate to the same value as the shock in question. Thus in continuous time the total response to the time varying shock $z_J(s)$, $0 \leqslant s \leqslant t$, is the integral of expression (20) over $[0,t]$:

$$y_J(t) = \int_0^t A_J^{-1} [e^{A_J(t-\tau)} - I] N_J z_J[\tau,t] d\tau . \qquad (21)$$

Now,

$$z_J[\tau,t] \equiv \frac{dz_J(s)}{ds}\Big|_{s=\tau} = \frac{dy_B(s)}{ds}\Big|_{s=\tau} = e^{A_B\tau} N_B z_B[0,t]$$

and hence,

$$y_J(t) = \int_0^t A_J^{-1} [e^{A_J(t-\tau)} - I] N_J e^{A_B\tau} d\tau N_B z_B[0,t] . \qquad (22)$$

Thus the accumulated Johansen response at time t can be expressed as a matrix:

$$(\int_0^t A_J^{-1} [e^{A_J(t-\tau)} - I] N_J e^{A_B\tau} d\tau) N_B$$

Now the first term appearing in (26) involves $C_J(t)$, and hence depends on assumption (15). If, however, (26) is evaluated at $t=t^*$, this assumption is not needed. The "stand alone" Johansen response is exact, and the only approximation involves the speed at which the Johansen model reacts to shocks being fed into the system by the Bergstrom model. Thus the Johansen model's response corrected for the macroeconomic environment as provided by the Bergstrom model is:

$$y_J(t^*) = C_J K z_J[0, t^*] + C_{J/B}(t^*) C_B(t^*) z_B[0,t^*] . \qquad (31)$$

It would be computationally convenient if the corrected Johansen response could be formulated purely in terms of shocks to the Johansen model. We therefore introduce the concept of an "as if" shock which we define as that shock to the Johansen exogenous variables which would produce a Johansen stand-alone response equivalent to the corrected response of the Johansen model to a shock to both models. Now given (30), (31) may be re-expressed as:

$$y_J(t^*) = C_J \begin{pmatrix} \overline{z_J y_B} \\ z_J z_B \\ z_J \tilde{x}_B \end{pmatrix} [0, t^*] , \qquad (32)$$

where $\overline{z_J y_B}$ is the "as if" shock, defined by:

$$\overline{z_J y_B} [0, t^*] = V(\beta^{t^*} - 1)^{-1} \{[A_B - (\ell n \beta) I]^{-1} [e^{A_B t^*} - \beta^{t^*} I][e^{A_B t^*} - I]^{-1} A_B - I\}$$

$$X C_B(t^*) z_B[0, t^*] . \qquad (33)$$

Consider, for example, the calculation of the response of the combined model to a shock to a variable exogenous to both models. The "stand-alone" Johansen response can be computed directly as $C_J K z_J[0,t^*]$. The response

The implications of this parameterisation can be seen by looking at the response to a sustained shock $z_J[0,t^*]$ of either (a) the rates of change of the proportional deviations of the Johansen model's endogenous variables, which follow a declining geometric progression:

$$\begin{aligned}\dot{y}_J(\tau) &= \beta^{\tau}\,\dot{y}_J(0) \quad , \quad 0 \leq \tau \leq t^* \, , \\ &= 0 \quad , \quad \tau > t^* \, ;\end{aligned}$$

or (b) the accumulated proportional deviations, which follow:

$$\begin{aligned}y_J(\tau) &= \frac{\beta^{\tau}-1}{\beta^{t^*}-1}\, y_J(t^*) \quad , \quad 0 \leq \tau \leq t^* \, , \\ &= y_J(t^*) \quad , \quad t > t^* \, .\end{aligned} \tag{29}$$

With specification (28), matrices such as A_J and $e^{A_J t}$ appearing in (27) are scalar matrices. Thus the integral component of expression (27) for $C_{J/B}(t)$ may be evaluated explicitly as:

$$C_{J/B}(t) = C_J\, V(\beta^{t^*}-1)^{-1} \{[A_B - (\ell_n\beta)I]^{-1} [e^{A_B t} - e^{(\ln\beta)t} I]\, [e^{A_B t} - I]^{-1} A_B - I\} \, , \tag{30}$$

(recall $C_J(t^*) \equiv C_J$) .

times a sustained shock $z_B[0,t]$ to the Bergstrom model. Equation (22) is the reduced form (adjusted for timing) of the recursive system (12*) - (13*). It is convenient to express (22) in structural form as:

$$\begin{aligned}y_J(t) &= C_{J/B}(t)\, y_B(t) \\ y_B(t) &= C_B(t)\, z_B[0,t] \, ,\end{aligned} \tag{23}$$

where $C_{J/B}(t) = \int_0^t A_J^{-1} [e^{A_J(t-\tau)} - I]\, N_J e^{A_B \tau} d\tau\, [e^{A_B t} - I]^{-1} A_B$.

The General Recursive Linkage

In the case where $z_J \neq y_B$ the combined model, uncorrected for timing, is (12) - (13). Again consider a sustained [0,t] shock to variables exogenous to the Bergstrom model. As in the previous sub-section the Bergstrom response is given by (17). Now, however, it is a subset of y_B, namely $y_B z_J$, which leads to a time varying shock $z_J y_B(s)$ to the Johansen model. Again this time path is represented as an integral of its derivatives. The typical derivative represents a sustained shock from τ to t, and by linearity the timing-corrected Johansen response can be represented as the integral of the responses to this overlapping sequence of sustained shocks. In addition, there is an avenue for a direct Johansen response to shocks other than those generated by the Bergstrom response. That is:

$$y_J(t) = \int_0^t A_J^{-1} [e^{A_J(t-\tau)} - I]\, N_J \begin{pmatrix} z_J y_B \\ 0 \\ 0 \end{pmatrix} [\tau,t] d\tau$$

$$+ \; C_J(t) \begin{pmatrix} 0 \\ z_J z_B \\ z_J \tilde{x}_J \end{pmatrix} [0,t] \; . \qquad (24)$$

Now, by definition, $z_J y_B \, [\tau,t] \; = \; \frac{d}{ds} \; z_J y_B(\tau) \, |_{s=\tau}$.

But:

$$\frac{d}{ds} \begin{pmatrix} y_B y_J \\ y_B z_J \\ y_B \tilde{x}_J \end{pmatrix} (s) \Bigg|_{s=\tau} \; = \; e^{A_B \tau} N_B z_B[0,t] \; .$$

Thus the complete timing - corrected Johansen response may be written:

$$y_J(t) \; = \; \int_0^t A_J^{-1} \, [e^{A_J(t-\tau)} - I] N_J V \, e^{A_B \tau} d\tau \; N_B z_B[0,t]$$

$$+ \; C_J(t) \; K \; z_J[0,t] \; , \qquad (25)$$

where $K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ and $V = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Thus the structural form generalisation of (23) for the general recursive link is:

$$y_J(t) \; = \; C_J(t) \; K \; z_J[0,t] + C_{J/B}(t) \; y_B(t)$$

$$y_B(t) \; = \; C_B(t) \; z_B[0,t] \; , \qquad (26)$$

where

$$C_{J/B}(t) \; = \; \int_0^t A_J^{-1} \, [e^{A_J(t-\tau)} - I] \; N_J V e^{A_B \tau} d\tau \, [e^{A_B t} - I]^{-1} A_B \; . \qquad (27)$$

A Useful Special Case

Previously it has been pointed out that a Bergstrom model typically allows the estimation of the dynamics of the model expressed in A_B and N_B and hence allows the derivation of C_B by (7). On the other hand a Johansen model typically allows only the direct estimation of C_J, a matrix which does not uniquely identify the pair A_J, N_J. In addition, the response (26), based on the definition of $C_{J/B}$ given in (27) would, in the generality used there, be prohibitively expensive to compute, requiring integral approximations. A convenient parameterisation which overcomes both of these problems is to choose A_J a scalar matrix, i.e.,[2]

$$A_J \; = \; (\ell n \; \beta) \; . \; I \quad , \quad 0 < \beta < 1 \; . \qquad (28)$$