IMPACT PROJECT

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THE THEORY OF LABOUR SUPPLY AND COMMODITY DEMAND WITH AN ENDOGENOUS MARGINAL WAGE RATE

by

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Phlips, Louis (1978), "The Demand for Leisure and Money," Econometrica, Vol. 46, No. 5, pp. 1025-1044.

Powell, Alan A. (1974), Empirical Analytics of Demand Systems, Lexington, Mass., D. C. Heath.

Powell, Alan A., Ashok Tulpulé and Richard J. Filmer (1977), "Commodity Specific Subsidies, Demand Patterns and the Incentive to Work," IMPACT Project Preliminary Working Paper No. BP-10, Industries Assistance Commission, Melbourne, Vic., Australia (mimeo).

Theil, Henri (1971), Principles of Econometrics, New York, John Wiley and Sons, Inc..

Tulpulé, Ashok (1978), "Empirical Estimation of Labour Supply Elasticities," IMPACT Project Preliminary Working Paper No. BP-12, Industries Assistance Commission, Melbourne, Vic., Australia (mimeo).

Woodland, A.D., and T.J. Wales (1979), "Labour Supply and Progressive Taxes," Review of Economic Studies, Vol. XLVI (1), No. 142, pp. 83-96.

Contents

REFERENCES

Abbott, M., and O. Ashenfelter (1976), "Labour Supply Commodity Demand and the Allocation of Time," Review of Economic Studies, Vol. XLIII (3), No. 135, pp. 389-411.

Barnett, William A. (1979), "The Joint Allocation of Leisure and Goods Expenditure," Econometrica, Vol. 47, No. 3, pp. 539-563.

Barten, Anton P. (1964), "Consumer Demand Functions Under Conditions of Almost Additive Preferences," Econometrica, Vol. 32, No. 1-2, pp. 1-38.

Betancourt, Roger R. (1973), "Household Behaviour in a Less Developed Country : An Econometric Analysis of Cross Section Data," Department of Economics, University of Maryland, College Park, Maryland (mimeo).

Burtless, Gary, and Jerry A. Hausman (1978), "The Effect of Taxation on Labour Supply : Evaluating the Gary Negative Income Tax Experiment," Journal of Political Economy, Vol. 86, No. 6, pp. 1103-1130.

Goldberger, Arthur S. (1967), "Functional Form and Utility : A Review of Consumer Demand Theory," University of Wisconsin, Social Systems Research Institute, Systems Formulation, Methodology and Policy Workshop Paper 6703 (mimeo).

Hicks, J. R. (1939), Value and Capital, Oxford, Oxford University Press.

Kiefer, Nicholas M. (1965), "Quadratic Utility Labour Supply and Community Demand," Industrial Relations Section, Princeton University, Princeton N.J. (mimeo).

Kiefer, Nicholas M. (1977), "A Bayesian Analysis of Commodity Demand and Labor Supply," International Economic Review, Vol. 18, No. 1, pp. 209-218.

Lluch, Constantino (1970), "The Extended Linear Expenditure System," University of Essex, Department of Economics, Discussion Paper No. 11 (mimeo).

Lluch, Constantino (1973), "The Extended Linear Expenditure System," European Economic Review, Vol. 4, pp. 21-32.

Lluch, Constantino, Alan A. Powell and Ross A. Williams (1977), Patterns in Household Demand and Saving, New York, Oxford University Press for the World Bank.

continued ...

7. CONCLUSION

Part-time employment is becoming much more common in many OECD countries than traditionally has been the case. The net after tax receipts for the marginal hour of labour supplied usually depends on the number of hours worked. This reflects the incidence of progressive income taxes and of penalty rates for overtime. The shadow valuation placed on broadly defined 'leisure' by a worker who declines an additional hour's work is equal to the net after tax receipts which would have been generated by the hour of work declined. This price is endogenous to the consumer/worker's time and budget allocation problems. In this paper the work of Hicks and Barten has been extended to endogenize the marginal wage. The theory therefore is of potential relevance to labour market behaviour in the 1980's. It provides a unified framework within which may be viewed the disparate empirical evidence on the response of the supply of labour to income related variables.

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1. INTRODUCTION

The seventies have seen a burgeoning of an empirical literature on labour-leisure choice grounded in the work on the systems approach to the estimation of demand systems which flowered in the sixties. Working with the Klein-Rubin utility function, Betancourt (1973) extended Lluch's ELES system (1970, 1973) to include the labour-leisure choice. Abbott and Ashenfelter (1976) analysed United States data on hours worked using both the Klein-Rubin and the indirect addilog utility functions (although, unlike Betancourt, they did not endogenize savings). Kiefer (1977) used a Bayesian approach to the estimation of a 7 commodity plus leisure system which imposed only classical restrictions on a Rotterdam model written in logarithmic differentials. He also explored the use of a quadratic indirect utility function as a second order Taylor approximation to an arbitrary nested indirect utility function in which Box/Cox transformations comprise the inner nests (Kiefer, 1975). Phlips (1978) has used an extended Klein-Rubin utility function, in which the real stock of money appears, to integrate the demand for commodities and leisure with the demand for money. Barnett (1979) has imbedded the commodity and leisure demand

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* I am grateful to my colleagues Ashok Tulpulé and Richard J. Filmer who worked with me on the paper which begat the current paper. Richard Filmer also drew my attention to some relevant literature which I had overlooked. I benefitted from the opportunities to present that earlier paper to a workshop at C.O.R.E., Louvain, in October 1978, and to present a draft of the current paper to the Institute of Labour Studies at the Flinders University of South Australia in September 1979. I am especially indebted to Peter B. Dixon, who checked the mathematics underlying the major results of the paper, and as well offered helpful comments. Tony Lawson and Ken Clements kindly gave detailed comments on the manuscript. The research was supported, in part, by the IMPACT Project. All errors are the exclusive responsibility of the author.

problem into a version of the Rotterdam model under relatively weak assumptions about preferences. Finally, recent empirical work using Betancourt's TELES model on Australian data has been reported by Tulpulé (1978).

In all of this work a maintained hypothesis has been that the after tax hourly wage rate is a constant. Labour income then becomes endogenous because the representative agent chooses the number of hours spent in paid employment so as to maximize utility. In relatively few papers is the usually assumed constancy of the marginal wage rate relaxed. Powell, Tulpulé and Filmer (1977) derived results on the basis of a Klein-Rubin utility function and an arbitrary after-tax marginal wage rate schedule. Burtless and Hausman (1978) start with a constant elasticity labour supply function, work back to the corresponding indirect utility function, and impose a budget constraint (not necessarily convex) which reflects discrete jumps in the (after tax) marginal wage rate. Woodland and Wales (1979) consider similar budget constraints with a CES utility function. It is the purpose of the present paper to generalize these results to include not only an arbitrary earnings-hours schedule, but also an arbitrary classical utility function. The treatment of the after tax hourly wage rate as a function of the number of hours worked enables a clearer discussion of the issues underlying the concept of a 'backward bending supply curve' for labour, and gives some insights into the likely effects on labour supply of increasing marginal tax rates, changes in overtime incentives, and reductions in standard hours of work.

The approach is to generalize Barten's matrix representation (1964) of Hicks' fundamental equation of value theory (1939). It is when taking differentials (with respect to the exogenous variables) of the first order conditions of the consumer's utility maximization problem that complications arise. The essential analytical trick is to recognize that the parameters of

Table 5

Effect of a 10 per cent Change in Parameters of the Earnings Schedule on Hours Worked and on Earnings in HYPOTHETICAL Numerical Example

Parameter	Parameter Value		Hours Worked (H)		Earnings (G)		Labour Supply 'Elasticity' with Respect to:	
	Before change	After change	Before change	After change (approx.)	Before change	After change	Total Earnings $\frac{\Delta H}{H} \Big/ \frac{\Delta G}{G}$ (approx.)	Average Wage[(1)] $\frac{\Delta H}{h} \Big/ \frac{\Delta[\frac{G}{H}]}{[\frac{G}{H}]}$ (approx.)
θ_1 (basic hourly wage) (10% increase)	4	4.4	41	40.63	164.4	178.93	-0.107	-0.097
			43	42.72	175.6	190.90	-0.078	-0.072
			45	44.76	190.0	206.02	-0.066	-0.062
			48	47.80	217.6	234.69	-0.055	-0.052
θ_2 (standard hours) (10% decrease)	40	36	41	42.58	164.4	187.64	0.286	0.401
			43	43.64	175.6	197.91	0.123	0.140
			45	45.29	190.0	215.68	0.051	0.054
			48	48.05	217.6	250.28	0.007	0.007
θ_3 (overtime progression) (10% increase)	0.8	0.88	41	41.04	164.4	164.64	0.685	2.175
			43	43.06	175.6	176.35	0.318	0.466
			45	45.05	190.0	191.46	0.158	0.188
			48	48.04	217.6	220.62	0.063	0.070

Source: Powell, Tulpulé and Filmer (1977) (modified).

1. Computed as follows: let elasticity in previous column = E . Figure in this column is E/(1-E) .

Some sense can now be made of the widely conflicting evidence on the subject of the response of labour supply to changes in earnings. Non-labour income, savings, or parametric changes of various sorts in the schedule representing labour income opportunities might be responsible. Alternatively (as has been argued elsewhere - - Powell, Tulpulé and Filmer (1977)), demographic and/or other compositional changes in aggregate time series data may affect the 'personality' of the representative agent as encapsulated in an aggregate set of utility parameters, $\{ \beta_0, \beta_1, \ldots, \beta_n; \quad \gamma_1, \gamma_2, \ldots, \gamma_n \}$. To illustrate but one of these sources of disparate (but theoretically plausible) responses, elasticities of hours worked with respect to changes in labour income are tabulated in Table 5 for the hypothetical agent whose supply behaviour is given in Table 4. Only the conditions of his employment opportunities are allowed to vary. Table 5 implies that a 'supply curve' attempting to relate H to G (or to G/H) could produce virtually any slope at all depending on the composition of the differential $d\underline{\underline{\Theta}}$ at the root cause of the variations.

the earnings-hours offer schedule are, from the point of view of the consumer, exogenous variables. Changes in these exogenous variables produce both a direct and an indirect response. In terms of elementary teaching concepts, one must keep track of not only shifts <u>of</u> the earnings-hours schedule, but also of shifts <u>along</u> it. The latter are to be regarded as the indirect effects of the parametric change.

The plan of the paper is as follows. In Section 2 notation is introduced, and the concept of an earnings-hours schedule is motivated. The consumer-worker's time and expenditure allocation problem and the associated first order conditions for utility maximization are written down. In the tradition of <u>Value and Capital</u>, the first order conditions are then differentiated with respect to the exogenous 'variables' of the system (which include the parameters of the earnings-hours schedule). This generates the fundamental matrix equations of commodity demand and labour supply under conditions in which the marginal wage rate is endogenous. The solution is discussed separately for variations in prices, non-labour income, and parametric changes in the earnings-hours schedule, in Sections 3, 4 and 5 respectively. In Section 6 an illustrative example based on the Klein-Rubin utility function is given. Concluding remarks are offered in Section 7.

2. THE FUNDAMENTAL MATRIX EQUATIONS

In order to keep the notation flexible, some duplication seems inevitable.

Let

$$(y_0, y_1, \ldots, y_n)$$

be the vector whose elements are respectively leisure consumed (broadly defined to mean any time not spent in paid work), and the elements of the commodity bundle purchased. Further, we will use

$$\underline{\underline{x}} = (x_1, \ldots, x_n)$$
$$\equiv (y_1, \ldots, y_n) .$$

By $\underline{\underline{p}}$ we shall denote the prices of commodities; that is

$$\underline{\underline{p}} = (p_1, \ldots, p_n) .$$

The total number of hours available per week (i.e., 168) will be denoted by T , and the number of hours spent in paid employment H . Then

$$y_0 \equiv T - H .$$

The earnings-hours schedule will be written

$$G(H, \underline{\underline{\Theta}}) \equiv G(H) ,$$

where $\underline{\underline{\Theta}}$ is the vector of m parameters characterizing the function G . The marginal wage rate (net, after tax) is denoted ψ ,

$$\psi = \frac{\partial G}{\partial H} = \psi(H, \Theta) \equiv \psi(H) .$$

An example of a ψ function involving just three parameters (namely, a basic net hourly wage rate Θ_1 , the number of hours in a standard working week Θ_2 , and the steepness Θ_3 of the overtime progression for work in excess of standard hours paid at penalty rates) is shown in Figure 1.

Changes in any one of the parameters Θ_1 , Θ_2 and Θ_3 induce both income and substitution effects. These have different signs. The net outcome depends, therefore, on their relative strengths. When the basic hourly wage Θ_1 is changed, it has a relatively large income effect, since it accrues over all hours worked up to H . At least in the case discussed the income effect of a change in Θ_1 is consequently large enough to outweigh the substitution effect. The latter is away from leisure (i.e., towards longer hours of work) due to the increase in the shadow price of leisure (namely, ψ).

A small cut in standard hours, however, does not have so large an income effect. What such a cut implies is that marginal hours previously rewarded at the standard rate Θ_1 are now paid at overtime penalty rates. In this case the substitution effect outweighs the income effect and (except in the case of those who are already working very large amounts of overtime) a cut in standard hours leads unambiguously to an increase in hours of labour supply offered.[1]

Increasing the steepness of the penalty rate structure for overtime has similar consequences. The income effect again only operates with respect to overtime hours and the substitution effect is likely to outweigh it, as in Table 4.

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1. To obtain the effects of a cut in standard hours with nominal weekly labour income held constant, add row 1 to the negative of row 2 in Table 4.

Table 4

Elasticities of Labour Hours Supplied with respect to the Parameters of an (after-tax) Earnings Schedule - a HYPOTHETICAL Numerical Example

Parameter	Elasticities of Hours Supplied for different values of H							
	$H \leqq 40$ (less than standard hours are worked)	Value of H (overtime is worked, i.e., $H > 40$) 41	42	43	44	45	48	50
Θ_1 (basic hourly wage)	-0.067	-0.091	-0.076	-0.066	-0.058	-0.053	-0.041	-0.035
Θ_2 (standard hours)	0	-0.393	-0.240	-0.154	-0.102	-0.066	-0.010	+0.008
Θ_3 (overtime progression)	0	0.010	0.013	0.014	0.013	0.012	0.009	0.007

Source : Powell, Tulpulé and Filmer (1977) (modified).

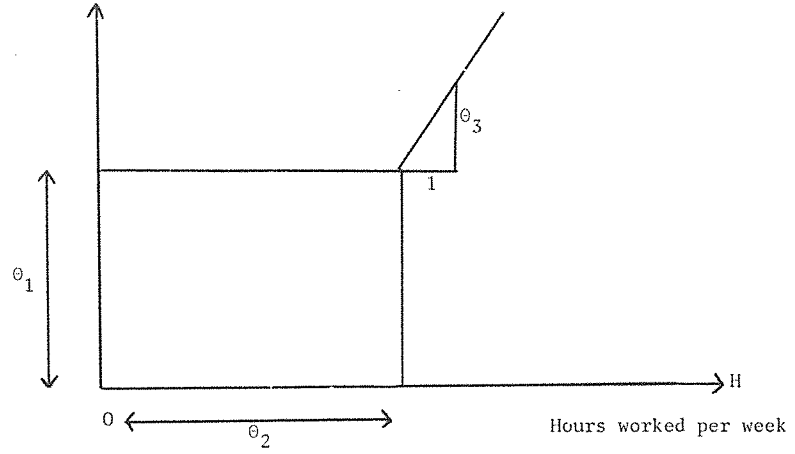

Figure 1 : A simple earnings-hours schedule with parameters Θ_1 (\$/hr, basic hourly wage rate), Θ_2 (hours, defining the standard working week), and Θ_3 (\$/hr/hr, steepness of the overtime progression). G(H) is the area under the curve.

Let the consumer who faces an earnings-hours offer schedule G(H) have utility function

(1) $u = u(y_1, y_2, \ldots, y_n; y_0)$,

in which u is strictly concave and twice differentiable everywhere in its domain (the non-negative Euclidean half-space of dimension $n+1$).

Unlike the situation shown in Figure 1, it is assumed that G has first and second derivatives with respect to H everywhere in its domain. Moreover, since functional differentials in G and ψ are to be generated by differentials in $\underline{\underline{\theta}}$, it is assumed that the first derivatives of G and ψ with respect to $\underline{\underline{\theta}}$ exist everywhere in the relevant region. Let α be the consumer's net after tax non-labour income, inclusive of transfer payments, less his personal savings. Along with $\underline{p}$ and $\underline{\underline{\theta}}$, α fills out the set of variables exogenous to the problem under study. The budget constraint facing the consumer is

$$\underset{\substack{\text{non-labour}\\\text{income -}\\\text{savings}}}{\alpha} + \underset{\substack{\text{labour}\\\text{income}}}{G(H)} = \underset{\substack{\text{total}\\\text{expenditure}}}{\underline{p}^T \underline{x}} \; ; \qquad (2)$$

and so the Lagrangean is

$$L = u + \lambda \; \{\alpha + G - \underline{p}^T \underline{x}\} \; . \qquad (3)$$

λ is a Lagrange multiplier whose interpretation is the marginal utility of an additional dollar of expenditure optimally spent.

Before writing down the first order conditions, it is helpful to introduce some further notation. Let

$$\frac{\delta u}{\delta y_j} = X_j(\underline{p}, \underline{\underline{\theta}}, \alpha) \qquad (j=0, 1, \ldots, n)$$

and

$$y_j = Q_j(\underline{p}, \underline{\underline{\theta}}, \alpha) \qquad (j=0, 1, \ldots, n)$$

respectively be the (indirect) marginal utility functions and the demand functions for commodities and leisure. Whenever a partial derivative is taken of a X_j or a Q_j, of λ, of ψ or of G, with all values of the exogenous variables $\{\underline{p}, \underline{\underline{\theta}}, \alpha\}$ held constant other than the variable with respect to which differentiation is proceeding, the derivatives will be denoted $\partial(\cdot) / \partial(\cdot)$. When a partial derivative is taken of G or ψ with

For many OECD countries $(-\omega)$ has been estimated to be somewhat under two.[1] In the example, $-\omega$ in (30) is set equal to 2.2727, yielding $\mu = -0.56G$. Thus the following information is now available for use in Table 3 :

$\theta_1 = 4$, $\theta_2 = 40$, $\theta_3 = 0.8$, $\beta_0 = 0.12$, $\mu = -0.56G$;
$G = 4H$ $[H \leq 40]$; $G = 160 + 4(H - 40) + 0.4(H - 40)^2$ $[H > 40]$

Using the above values in Table 3 yields Table 4.

The individuals whose responsiveness to marginal changes in conditions of employment is shown in Table 4 are by assumption of the same occupational type and the same level of skill and experience. They face identical earnings-hours schedules (that is, equal employment opportunities). These individuals face the same set of commodity prices and by construction have identical non-labour income and savings behaviour. The reasons that different individuals choose to work different numbers of hours is solely explained by the values of their utility parameters $(\underline{\underline{\beta}}, \underline{\underline{\gamma}})$.

The leading element of the first row of Table 4 is the standard 'backward bending supply curve of labour'. Indeed, results in the first column of Table 4 are obtainable by endogenizing labour income without endogenizing the marginal wage rate. The new insights obtainable from the theoretical sections of this paper relate to the behaviour of those who choose to work overtime (in the example, $H > 40$). The most dramatic results are those of rows 2 and 3 which show respectively the elasticities of hours supplied with respect to an <u>increase</u>[2] in standard hours with a fixed overtime progression, and with respect to an increase in the overtime progression at fixed standard hours.

1. Lluch, Powell and Williams (1977), p.75.
2. To obtain the effects of a cut in standard hours, reverse the signs in row 2 of Table 4.

Estimates of β_0 are available from national time-series data for the U.S. (Abbott and Ashenfelter (1976)) and for Australia (Tulpulé (1978)). Estimates of (i), (iii) and (v) for a typical worker might be obtained from survey data, while (ii) is a matter of public record. To illustrate the methods a hypothetical example (based on Powell, Tulpulé and Filmer (1977)) is now constructed.

Suppose that a worker's weekly standard hours are 40 and that he faces the following after tax earnings schedulue :

G(H) { he can earn \$4 per hour after tax for the first 40 hours of work; thereafter the net marginal wage rate rises at 80 cents per hour per hour.

That is, Θ_1 = \$4 per hour, Θ_2 = 40 hours/week, and Θ_3 = \$0.8 per hour per hour. The worker's elasticity of labour supply with respect to the parameters of G can be calculated provided that we are able to put values on β_0 and μ. For illustrative purposes, Abbott and Ashenfelter's (1976) estimate of β_0 is taken; namely, $\beta_0 = 0.12$. To simplify the example it is assumed that the worker has no asset or transfer income, and that he does not save. Thus $\alpha = 0$. To estimate μ use is made of the relationship which prevails in the linear expenditure system (LES) between the 'cost of subsistence' $(\sum_{i=1}^{n} p_i \gamma_i)$ and ω, the Frisch 'parameter'.[1] In the LES the Frisch parameter[2] is

$$(30) \qquad -\omega = \frac{G}{\mu+G} .$$

1. ω strictly speaking is a variable. In empirical time series work based on the LES it is common to evaluate ω at sample mid-points, and to attribute variations in ω about this value to shortcomings in the specification. Country-specific estimates of ω obtained in this way vary systematically across the development spectrum. See Lluch, Powell and Williams (1977).

2. Goldberger (1967).

respect to a parameter Θ_i but at a fixed value of H, derivatives will be written $\partial^*(\cdot) / \partial^*(\cdot)$. The ordinary derivatives of G and ψ with respect to H at fixed values of $\underline{\Theta}$ are denoted ψ and ψ' respectively. Finally, conventional partial derivatives of the (direct) utility function and of the Lagrangean will be written $\delta(\cdot) / \delta(\cdot)$.

The first order conditions are:

$$(3.1) \qquad \frac{\delta L}{\delta y_i} = \frac{\delta u}{\delta y_i} - \lambda p_i = 0 \qquad (i=1, \ldots, n) ,$$

$$(3.2) \qquad \frac{\delta L}{\delta y_0} = \frac{\delta u}{\delta y_0} - \lambda \psi = 0 ,$$

$$(3.3) \qquad \alpha + G - \underline{p}^T \underline{x} = 0$$

Let ξ be any exogenous variable; i.e., a commodity price, a parameter of the earnings-hours schedule, or non-labour income minus savings. Then[1]

$$(4) \qquad \frac{\partial x_i}{\partial \xi} = \frac{\partial \left[\frac{\delta u}{\delta y_i}\right]}{\partial \xi} = \sum_{\ell=0}^{n} \frac{\delta^2 u}{\delta y_i \delta y_\ell} \frac{\partial Q_\ell}{\partial \xi} \qquad (i=0, 1, \ldots, n).$$

Let δ_{ij} be Kronecker's delta. Then, differentiating the remaining terms in (3.1) - (3.3) with respect to the j^{th} commodity price, we obtain

$$(5.1) \qquad \partial(\lambda p_i) / \partial p_j = \delta_{ij} + p_i \frac{\partial \lambda}{\partial p_j} \qquad (i,j=1, \ldots, n) ,$$

$$(5.2) \qquad \partial(\lambda\psi) / \partial p_j = \psi \frac{\partial \lambda}{\partial p_j} - \lambda\psi' \frac{\partial Q_0}{\partial p_j} \qquad (j=1, \ldots, n),$$

1. See, e.g., Powell (1974), p.7-8.

$$(5.3) \qquad \partial(\underline{p}^T\underline{x}) / \partial p_j = x_j + \sum_{\ell=1}^{n} p_\ell \frac{\partial Q_\ell}{\partial p_j} \qquad (j=1, \ldots, n),$$

$$(5.4) \qquad \partial G / \partial p_j = - \psi \frac{\partial Q_0}{\partial p_j} \qquad (j=1, \ldots, n),$$

$$(5.5) \qquad \partial \alpha / \partial p_j = 0 \qquad (j=1, \ldots, n).$$

In (5.2) and (5.4) the terms $-\lambda\psi' \, \partial Q_0/\partial p_j$ and $\psi \, \partial Q_0/\partial p_j$ represent the indirect effects of the changes in commodity prices; that is, the effects on ψ and G respectively due to the movement along a fixed offer curve of the number of hours worked.

Next, the first order conditions (3.1-3) are differentiated with respect to the parameters $\underline{\Theta}$ of the earnings-hours schedules. As before, we proceed term by term.

$$(6.1) \qquad \partial(\lambda p_i)/\partial\Theta_k = p_i \frac{\partial\lambda}{\partial\Theta_k} \quad (i=1, \ldots, n),$$

$$(6.2) \qquad \partial(\lambda\psi)/\partial\Theta_k = \psi \frac{\partial\lambda}{\partial\Theta_k} + \lambda \frac{\partial^*\psi}{\partial^*\Theta_k} - \lambda\psi' \frac{\partial Q_0}{\partial\Theta_k},$$

$$(6.3) \qquad \partial(\underline{p}^T\underline{x})/\partial\Theta_k = \sum_{\ell=1}^{n} p_\ell \frac{\partial Q_\ell}{\partial\Theta_k},$$

$$(6.4) \qquad \partial G/\partial\Theta_k = \frac{\partial^* G}{\partial^*\Theta_k} - \psi \frac{\partial Q_0}{\partial\Theta_k},$$

$$(6.5) \qquad \partial\alpha/\partial\Theta_k = 0 .$$

(6.1)–(6.5) hold for $(k=1, \ldots, m)$.

In (6.2) and (6.4) the terms involving the $(\partial^*(\cdot)/\partial^*(\cdot))$ operator correspond to the direct effects (on the marginal wage ψ and on total labour earnings G respectively) of the small parametric change in the

Table 3

Elasticities of Labour Hours Supplied with respect to the Parameters of a Particular Earnings Schedule when Utility is Klein-Rubin

Parameter	Elasticities of Hours Supplied for different values of H	
	$H < \Theta_2$ (less than standard hours are worked)	$H > \Theta_2$ (overtime is worked)
Θ_1 (basic hourly wage)	$\frac{\beta_0 \mu}{G}$	$\frac{\Theta_1 \left[\frac{G+\mu}{\psi} - H\right]}{H\left[\frac{\psi}{\beta_0} - \frac{\Theta_3}{\psi}(G+\mu)\right]}$
Θ_3 (overtime progression)	0	$\frac{\Theta_3 \left[\frac{G+\mu}{\psi}(H-\Theta_2) - \frac{(H-\Theta_2)^2}{2}\right]}{H\left[\frac{\psi}{\beta_0} - \frac{\Theta_3}{\psi}(G+\mu)\right]}$
Θ_2 (standard hours)	0	$\frac{\Theta_2\Theta_3 \left[-\frac{G+\mu}{\psi} + (H-\Theta_2)\right]}{H\left[\frac{\psi}{\beta_0} - \frac{\Theta_3}{\psi}(G+\mu)\right]}$

Source: Powell, Tulpulé and Filmer (1977).

Notes : μ is $\alpha - \sum_{i=1}^{n} p_i \gamma_i$. The term $1/[\frac{\psi}{\beta_0} - \frac{\Theta_3}{\psi}(G+\mu)]$ has interpretation $y_{0\alpha}$.

Table 2

Derivatives of Endogenous Variables with Respect to the Parameters $\{\Theta_1, \ldots, \Theta_m\}$ of an Arbitrary Earnings-Hours Schedule, G, when the Utility Function is Klein-Rubin

Endogenous Variables	Elasticity with respect to the k^{th} parameter, Θ_k, of the total earnings schedule, G (b)(c)
Demand for Commodities, x_i ($i = 1, \ldots, n$; $k = 1, \ldots, m$)	$\frac{\partial x_i}{\partial \Theta_k} = \frac{\beta_i}{p_i}\left\{\frac{\psi'\beta_0(G+\mu) - \psi^2}{\psi'\beta_0(G+\mu)-(1-\beta_0)\psi^2}\right\}\left\{\frac{\partial^* G}{\partial^*\Theta_k} - \frac{\psi\,\beta_0\,(G+\mu)}{\psi'\beta_0(G+\mu)-(1-\beta_0)\psi^2}\frac{\partial^*\psi}{\partial^*\Theta_k}\right\}$
Marginal Utility of Money, λ ($k=1, \ldots, m$)	$\frac{\partial\lambda}{\partial\Theta_k} = \frac{(1-\beta_0)[\psi'\beta_0(G+\mu) - (1-\beta_0)\psi^2]}{(G+\mu)^2[\psi^2 - \psi'\beta_0(G+\mu)]}\frac{\partial^* G}{\partial^*\Theta_k} - \frac{(1-\beta_0)}{(G+\mu)}\frac{1}{\left[\frac{\psi}{\beta_0} - \frac{\psi'}{\psi}(G+\mu)\right]}\frac{\partial^*\psi}{\partial^*\Theta_k}$
Demand for Leisure(a), y_0 ($k=1, \ldots, m$)	$\frac{\partial y_0}{\partial\Theta_k} = \frac{\frac{\partial^* G}{\partial^*\Theta_k} - \frac{(G+\mu)}{\psi}\frac{\partial^*\psi}{\partial^*\Theta_k}}{\left[\frac{\psi}{\beta_0} - \frac{\psi'}{\psi}(G+\mu)\right]}$.

Source: Powell, Tulpulé and Filmer (1977) (modified).

(a) For hours of work supplied, reverse sign.

(b) In the formulae, μ is $(\alpha - \sum_{i=1}^{n} p_i y_i)$.

(c) The expression $1/\left[\frac{\psi}{\beta_0} - \frac{\psi'}{\psi}(G+\mu)\right]$, appearing in rows 2 and 3, has the interpretation $\partial y_0 / \partial\alpha$.

earnings-hours schedule. These terms are to be identified with the shift of the schedule G at given hours H. In the same equations the terms involving $(\partial Q_0/\partial\Theta_k)$ correspond to the indirect effects of the parametric change, that is, they correspond to changes in ψ and G respectively caused by the change in hours worked H which is induced by the change in the offered conditions of employment. The terms involving $(\partial Q_0/\partial\Theta_k)$ therefore represent movements along the earnings-hours offer schedule G.

Finally the first order conditions are differentiated with respect to the remaining exogenous variable, namely non-labour income spent, α . Because G and ψ are not functions of α , only indirect effects of the change in α on ψ and G appear in the equations[1]:

(7.1) $$\partial(\lambda p_i)/\partial\alpha = p_i \frac{\partial\lambda}{\partial\alpha} \qquad (i=1,\ldots,n) ,$$

(7.2) $$\partial(\lambda\psi)/\partial\alpha = \psi\frac{\partial\lambda}{\partial\alpha} - \lambda\psi'\frac{\partial Q_0}{\partial\alpha} ,$$

(7.3) $$\partial(\underline{p}^T\underline{x})/\partial\alpha = \sum_{\ell=1}^{n} p_\ell \frac{\partial Q_\ell}{\partial\alpha} ,$$

(7.4) $$\partial G/\partial\alpha = -\psi\frac{\partial Q_o}{\partial\alpha} ,$$

(7.5) $$\partial(\alpha)/\partial\alpha = 1 .$$

1. If the tax and transfer arrangements are such that G and ψ are functions of α , then (7.2) and (7.4) need additional right hand terms added to them; namely $\frac{\partial\psi}{\partial\alpha}$ and $\frac{\partial G}{\partial\alpha}$ respectively, where these derivatives are evaluated at fixed values of H and $\underline{\Theta}$.

Using (4), equation sets (5), (6) and (7) may be combined into a single matrix equation giving the slopes of all the endogenous variables with respect to all of the exogenous variables. Some further notation will assist in condensing the representation of the system. Let $\mathbf{U}$ be the n×n sub-Hessian of u corresponding to commodities, and $\mathbf{u}_0$ be the column of the Hessian corresponding to leisure. By $\mathbf{X}_p$ and $\mathbf{X}_\Theta$ respectively denote the (n×n) and (n×m) matrices with typical elements $(\partial Q_i/\partial p_j)$ and $(\partial Q_i/\partial \Theta_k)$ (where $i,j=1, \ldots, n;\ k=1, \ldots, m$). By $\mathbf{x}_\alpha$ denote the n-vector of commodity demand slopes with respect to α. Let $\boldsymbol{\lambda}_p$, $\boldsymbol{\lambda}_\Theta$ respectively denote the n- and m-vectors with typical elements $(\partial\lambda/\partial p_j)$ and $(\partial\lambda/\partial\Theta_k)$. By $\mathbf{G}_\Theta$ and $\boldsymbol{\psi}_\Theta$ respectively denote the m-vectors with typical elements $(\partial^* G/\partial^*\Theta_k)$ and $(\partial^*\psi/\partial^*\Theta_k)$, while by $\mathbf{y}_{0p}$ and $\mathbf{y}_{00}$ denote the n- and m-vectors whose elements are the slopes $(\partial Q_0/\partial p_j)$ and $(\partial Q_0/\partial\Theta_k)$ of leisure demand with respect to prices, and parameters of G, respectively. $\mathbf{I}_n$ is an n-dimensional identity matrix. In this notation the entire system generated from the differentiation of the first order conditions can be written :

$$\begin{bmatrix} 0 & \mathbf{p}^T & \psi \\ \mathbf{p} & \mathbf{U} & \mathbf{u}_0 \\ \psi & \mathbf{u}_0^T & u_{00}+\lambda\psi' \end{bmatrix} \begin{bmatrix} -\boldsymbol{\lambda}_p & -\boldsymbol{\lambda}_\Theta & -\frac{\partial\lambda}{\partial\alpha} \\ \mathbf{X}_p & \mathbf{X}_\Theta & \mathbf{x}_\alpha \\ (\mathbf{y}_{0p})^T & (\mathbf{y}_{0\Theta})^T & \frac{\partial Q_0}{\partial\alpha} \end{bmatrix} = \begin{bmatrix} -\mathbf{x}^T & \mathbf{G}_\Theta^T & 1 \\ \lambda\,\mathbf{I}_n & \underset{n\times m}{\mathbf{0}} & \underset{n\times 1}{\mathbf{0}} \\ \underset{1\times n}{\mathbf{0}^T} & \lambda\,\boldsymbol{\psi}_\Theta^T & 0 \end{bmatrix} . \qquad (8)$$

Equation (8) is the fundamental matrix equation of demand theory when the marginal wage rate is endogenous. For shorthand (8) will be written

beyond standard hours Θ_2 depend on the number of hours worked H. Although in a particular job the incline in Figure 1 might actually be a step function, the exact loci of the steps would vary slightly among employers. In a longer run context, therefore, from an employee's point of view, the opportunity set is defined by the envelope of such curves, to which Figure 1 is an approximation.

Given Klein-Rubin utility, but without yet restricting the form of G, the theory developed in Sections 2 and 3 yields the results shown in Table 2. If the earnings-hours schedule is as Figure 1, labour supply responses are further specialized to those shown in Table 3.

In the case of a Klein-Rubin utility function, the amount of data needed to quantify the elasticities given in Table 3 is relatively small. To estimate the supply of hours response of a typical worker, the following data are sufficient:

(i) the number of hours he currently works, the availability of overtime, and the conditions on which it is available;

(ii) the personal income tax schedule;

(iii) his non-labour income less savings ;

(iv) his marginal leisure preference β_0 ;

(v) his average propensity to save ;

and

(vi) Frisch's parameter ω (the elasticity with respect to total expenditure of the marginal utility of total expenditure).

Data on (iv) can be obtained from econometric demand studies; a great deal of empirical evidence on (vi) is already available in the literature.[1]

1. See especially Lluch, Powell and Williams (1977), Chapter 4.

6. ILLUSTRATIVE APPLICATION : KLEIN-RUBIN UTILITY WITH OVERTIME LOADINGS

The theory developed above is rich with implications for labour supply behaviour. To illustrate the use of the model attention is focussed on a phenomenon which has lead to a voluminous, but inconclusive empirical literature; namely, that concerning the so-called backward-bending supply curve for labour. In common with the 'new home economics'[1] the utility function is assumed to be strongly separable between 'leisure' (i.e., non-market activity) and paid work; more specifically, the utility function is assumed to be directly additive in all its arguments, and of the Klein-Rubin (or Stone-Geary) functional form:

$$u = \sum_{i=0}^{n} \beta_i \, \ell n(y_i - \gamma_i) \quad . \tag{29}$$

The earnings-hours schedule is assumed to be as in Figure 1.[2] In an economy in which a proportional income tax operates and in which penalty rates are paid to skilled workers for overtime and for work on weekends, etc., Figure 1 captures the essential idea that rates of hourly pay

1. See the special issue of the Journal of Political Economy, Vol.82, No.2, Part II (March/April 1974) for a collection of papers dealing with the foundations of the new home economics.
2. A functional form maintaining differentiability at Θ_2 , for which I am indebted to John Sutton, is

$$\text{(i)} \quad \psi(H) = \sqrt{(\varepsilon\Theta_1)^2 + \left[\frac{\Theta_3}{2}(H-\Theta_2)\right]^2} + \frac{\Theta_3}{2}(H-\Theta_2) + \Theta_1 \quad .$$

For ε chosen small, the above equation approximates the graph of Figure 1, namely

$$\text{(ii)} \quad \psi(H) = \left|\frac{\Theta_3}{2}(H-\Theta_2)\right| + \frac{\Theta_3}{2}(H-\Theta_2) + \Theta_1 \quad ,$$

as well as desired. In the text the latter equation is taken for ψ ; the (non-existent) derivative of ψ at Θ_2 is simply side-stepped. The alternative -- the use of (i) -- leads to a derivative which is arbitrarily determined by ε .

$$\underline{\underline{V}}\,\underline{\underline{Y}} = \underline{\underline{Z}} \quad , \tag{9}$$

a system of $(n + 2)\ (n + m + 1)$ equations in a like number of unknowns. The latter are the slopes of the $(n + 2)$ endogenous variables (n commodities, leisure, and λ) with respect to the $(n + m + 1)$ exogenous variables, $\underline{\underline{p}}$, $\underline{\underline{\Theta}}$ and α .

When (9) is solved for $\underline{\underline{Y}}$, we obtain the fundamental matrix equation of value theory when the marginal wage is endogenous. Whilst it is possible that $\underline{\underline{V}}$ might be singular, such an occurrence would be unlikely in practice.[1] Another difficulty relates to the possible break down of second order conditions. In these cases the consumer/worker opts for corner solutions $H = 0$ or $H = T$. Such a pathological possibility is illustrated in Figure 2.[2] The relevance of corner solutions is ultimately an empirical matter. From this point it is assumed that a solution for H interior to $[0, T]$ exists, and that $\underline{\underline{V}}$ is of full rank.

1. Strong assumptions which are sufficient to ensure uniqueness and that the second order conditions are not violated are as follows: That $\underline{\underline{U}}$ is negative definite with u strongly separable between commodities and leisure ($\underline{\underline{u}}_0 = \underline{\underline{0}}$) , that ψ' is non positive and that ψ is positive - - see equation (15). In the case $\psi' > 0$ (and again assuming $\underline{\underline{u}}_0 = \underline{\underline{0}}$) , multiple solutions arise when the (non-convex) budget constraint and the relevant indifference curve are contiguous over an interval, rather than at a point. The odds against the utility and budget surfaces exhibiting this property must be very long.

2. The author is grateful to Daniel Weiserbs for instructive comment on this point.

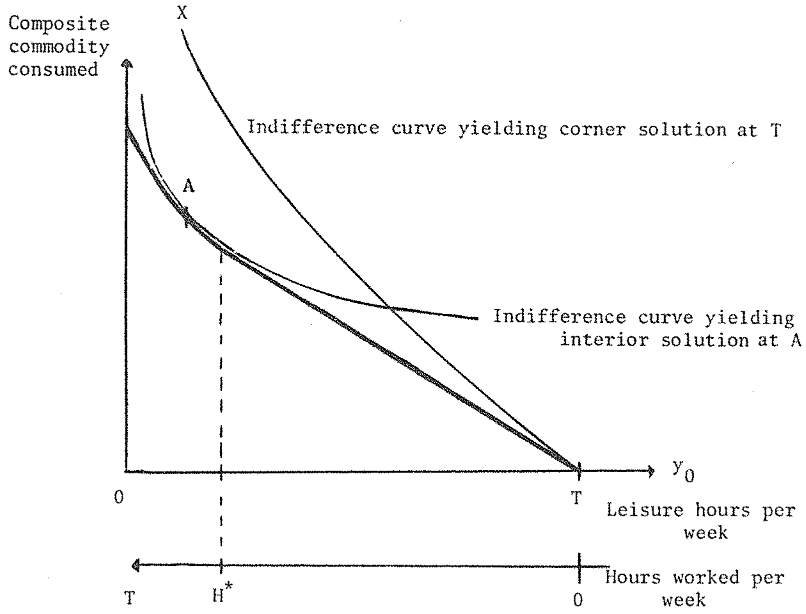

Figure 2 : The heavy line is the budget constraint of a consumer/worker who faces a ψ schedule which is flat from 0 to H^* hours, but which then rises. The relative convexity of this line and of the indifference map determines whether the solution point is in the interior of $[0, T]$ or at a corner.

The first term on the right of (27) is the direct income effect of the parametric change in earnings opportunities, that is, the response in commodity demands which would occur if the differentials in $(\Theta_1, \ldots, \Theta_m)$ did NOT simultaneously

(a) change the relative prices of commodities and leisure ; and

(b) cause the number of hours of worked and hence income to change.

Various partitions of the second right hand term of (27) are possible.

The final derivative of interest is

$$(28) \qquad \underline{\underline{y}}_{0\Theta} = y_{0\alpha}\left[\underline{\underline{G}}_{\Theta} - \frac{\lambda}{\lambda_{\alpha}}\, y_{0\alpha}\, \underline{\underline{\psi}}_{\Theta}\right] + \frac{\lambda \underline{\underline{\psi}}_{\Theta}}{u_{00} + \lambda\psi' - \underline{\underline{u}}_0^T\, \underline{\underline{U}}^{-1}\, \underline{\underline{u}}_0} .$$

Again a direct income effect, $y_{0\alpha}\,\underline{\underline{G}}_{\Theta}$, is easily identified. The remaining terms jointly contain an indirect income effect (due to the change in hours worked) plus specific and general substitution effects. These are more easily discussed in the context of a particular utility function and G schedule than in abstract.

5. DEMAND RESPONSES TO CHANGES IN EARNINGS OPPORTUNITIES

The remaining elements of the solution to (8) are now discussed. First,

$$(25) \qquad \boldsymbol{\lambda}_{\Theta} = -\left(\lambda y_{0\alpha}\, \boldsymbol{\psi}_{\Theta}^{T} - \lambda_{\alpha}\, \mathbf{G}_{\Theta}^{T}\right)$$

gives the responses of the marginal utility of money with respect to parametric changes in the earnings-hours schedule. In many applications corresponding elements of $\boldsymbol{\psi}_{\Theta}$ and $\mathbf{G}_{\Theta}$ will be similarly signed (i.e., a parametric change which raises total earnings will raise the marginal wage rate as well). Since usually $\lambda>0$ (non-satiation axiom), $y_{0\alpha}>0$ (normality of leisure) and $\lambda_{\alpha}<0$ (diminishing marginal utility), ambiguities of sign for elements of $\boldsymbol{\lambda}_{\Theta}$ seem unlikely to arise. In the special case in which the marginal wage is constant, (25) reduces to

$$(26) \qquad \boldsymbol{\lambda}_{\Theta} = -\left(\lambda y_{0\alpha} - \lambda_{\alpha} H\right) ,$$

where $\Theta \equiv \psi$ is the marginal wage (the sole parameter of G). This result closely parallels (19).

The slopes of commodity demands with respect to the parameters of the earnings-hours schedule are :

$$(27) \qquad \mathbf{X}_{\Theta} = \mathbf{x}_{\alpha}\, \mathbf{G}_{\Theta} - \frac{\lambda \left[\mathbf{x}_{\alpha}\left(\psi - \mathbf{p}^{T}\mathbf{U}^{-1}\mathbf{u}_{0}\right) + \mathbf{U}^{-1}\mathbf{u}_{0}\right]\boldsymbol{\psi}_{\Theta}^{T}}{u_{00} + \lambda\psi' - \mathbf{u}_{0}^{T}\mathbf{U}^{-1}\mathbf{u}_{0}}$$

The partitioned inversion of $\mathbf{V}$ is carried out in two steps. The first step uses Barten's (1964) result for the inverse of a (symmetric) matrix with a leading zero; namely

$$(10.1) \qquad \left[\begin{array}{c|c} 0 & \boldsymbol{\Pi}^{T} \\ \hline \boldsymbol{\Pi} & \mathbf{W} \end{array}\right] = \frac{1}{(\det)}\left[\begin{array}{c|c} -1 & \boldsymbol{\Pi}^{T}\mathbf{W}^{-1} \\ \hline \mathbf{W}^{-1}\boldsymbol{\Pi} & (\det)\mathbf{W}^{-1} - \mathbf{W}^{-1}\boldsymbol{\Pi}\boldsymbol{\Pi}^{T}\mathbf{W}^{-1} \end{array}\right]$$

in which the determinant of the left hand matrix is written (det), and

$$(10.2) \qquad (\det) = \boldsymbol{\Pi}^{T}\mathbf{W}^{-1}\boldsymbol{\Pi} .$$

We can apply (10.1) to $\mathbf{V}$ by defining

$$(11) \qquad \mathbf{W} = \left[\begin{array}{c|c} \mathbf{U} & \mathbf{u}_{0} \\ \hline \mathbf{u}_{0}^{T} & u_{00} + \lambda\psi' \end{array}\right]$$

and

$$(12) \qquad \boldsymbol{\Pi}^{T} = (p_{1}, \ldots, p_{n}; \psi) .$$

Using (10.1), and working in the $(1, 3)^{th}$ block of (8) we obtain

$$(13) \qquad \partial\lambda/\partial\alpha = (\det)^{-1} = 1/(\boldsymbol{\Pi}^{T}\mathbf{W}^{-1}\boldsymbol{\Pi}) .$$

Now working in the $(2, 3)^{th}$ and $(3, 3)^{th}$ blocks of (8), from (10.1) we obtain

$$(14) \qquad \left[\begin{array}{c} \mathbf{x}_{\alpha} \\ \hline \dfrac{\partial Q_{0}}{\partial\alpha} \end{array}\right] = \frac{\partial\lambda}{\partial\alpha}\, \mathbf{W}^{-1}\boldsymbol{\Pi} .$$

Using ordinary methods of partitioned inversion (e.g., Theil (1971), p. 18), the inverse of $\mathbf{W}$ is found to be

$$(15) \qquad \mathbf{W}^{-1} = \left[\begin{array}{c|c} \left[\mathbf{U} - \dfrac{\mathbf{u}_0\mathbf{u}_0^T}{u_{00} + \lambda\psi'}\right]^{-1} & \dfrac{-\mathbf{U}^{-1}\mathbf{u}_0}{(u_{00} + \lambda\psi' - \mathbf{u}_0^T\mathbf{U}^{-1}\mathbf{u}_0)} \\ \hline \dfrac{-\mathbf{u}_0^T\mathbf{U}^{-1}}{(u_{00} + \lambda\psi' - \mathbf{u}_0^T\mathbf{U}^{-1}\mathbf{u}_0)} & \dfrac{1}{(u_{00} + \lambda\psi' - \mathbf{u}_0^T\mathbf{U}^{-1}\mathbf{u}_0)} \end{array}\right]$$

An alternative expression for the non-diagonal blocks in (15) is

$$(16) \qquad -\mathbf{U}^{-1}\mathbf{u}_0 / (u_{00} + \lambda\psi' - \mathbf{u}_0^T\mathbf{U}^{-1}\mathbf{u}_0)$$

$$= \frac{-1}{u_{00} + \lambda\psi'}\left[\mathbf{U} - \mathbf{u}_0\mathbf{u}_0^T / (u_{00} + \lambda\psi')\right]^{-1}\mathbf{u}_0 \ .$$

Using (14), (15) and (16), we obtain

$$(17.1) \qquad \mathbf{x}_\alpha = \frac{\partial\lambda}{\partial\alpha}\left[\mathbf{U} - \frac{\mathbf{u}_0\mathbf{u}_0^T}{u_{00} + \lambda\psi'}\right]^{-1}\left[\mathbf{p} - \frac{\psi\mathbf{u}_0}{(u_{00} + \lambda\psi')}\right] ,$$

and

$$(17.2) \qquad y_{0\alpha} = \frac{\partial Q_0}{\partial\alpha} = \frac{\partial\lambda}{\partial\alpha}\ \frac{\psi - \mathbf{u}_0^T\mathbf{U}^{-1}\mathbf{p}}{u_{00} + \lambda\psi' - \mathbf{u}_0^T\mathbf{U}^{-1}\mathbf{u}_0} \ .$$

4. RESPONSES TO EXOGENOUS INCOME

The third (super) column of $\mathbf{Y}$ in (8) is now examined. The first element is the slope of λ, the marginal utility of money, with respect to α, non labour income minus savings. This we have already derived in (18). Notice that (18) is never Barten-isomorphic. In the case of strong separability between leisure and commodities, (18) reduces to

$$(23) \qquad \lambda_\alpha = \left[\mathbf{p}^T\mathbf{U}^{-1}\mathbf{p} + \frac{\psi^2}{u_{00} + \lambda\psi'}\right]^{-1} .$$

In the commodities only case, only the first of the two right hand terms in (23) appears.

The next item in (8)'s solution is $\mathbf{x}_\alpha$, the vector of non-labour income derivatives of commodity demands. This has also been derived above (see eqn (17.1)). If $\mathbf{u}_0$ is put equal to $\mathbf{0}$ in (17.1) we obtain

$$(24) \qquad \mathbf{x}_\alpha = \lambda_\alpha\,\mathbf{U}^{-1}\mathbf{p} \ .$$

This is the result in the commodities only case provided λ_α is reinterpreted as the slope of the marginal utility of money with respect to (exogenous) total spending. Thus under strong separability between commodities and leisure, (17.1) is Barten-isomorphic.

The final derivative with respect to α is the derivative of leisure demand (the negative of the derivative of labour supply). This is given above in (17.2).

terms on the right of (21) respectively are the income and general substitution effects. The second term on the right of (21) corresponds to the 'specific' substitution effect. It is in this term that the differences from the standard case are most visible. Equation (21) is Barten-isomorphic if $\underline{\underline{u}}_0 \equiv \underline{\underline{0}}$; that is, if commodities are strongly separable from leisure. In such a case specific interaction of individual commodities with leisure is ruled out. In particular, complementarity between leisure and any particular commodity is ruled out when $\underline{\underline{u}}_0$ vanishes. For empirical work (as in e.g., the "new home economics") which maintains the hypothesis $\underline{\underline{u}}_0 = \underline{\underline{0}}$, it is obviously important that the commodity classification be a broad one that does not quarantine all time-intensive market goods (e.g., motel vacation expenses) into a single item of the consumer's budget.

The final derivative with respect to commodity prices in the solution to (8) is

$$\underline{\underline{y}}_{0p} = \frac{\partial y_0}{\partial \underline{\underline{p}}} = - \frac{\partial H}{\partial \underline{\underline{p}}}$$

$$(22) \qquad = - y_{0\alpha} \underline{\underline{x}}^T - \frac{\lambda \, \underline{\underline{u}}_0^T}{u_{00} + \lambda\psi'} \left[\underline{\underline{U}} - \frac{\underline{\underline{u}}_0 \, \underline{\underline{u}}_0^T}{u_{00} + \lambda\psi'} \right]^{-1}$$

$$- \frac{\lambda}{\lambda_\alpha} y_{0\alpha} \underline{\underline{x}}_\alpha^T .$$

Various partitions of (22) into income and various types of substitution effects would be possible. This is because of the greater variety of income related exogenous variables: a differential $d\underline{\underline{p}}$ in prices might be sterilized by a non-labour income differential $d\alpha$, or by a suitable (n-dimensional) differential $d\underline{\underline{\Theta}}$ in the parameters of the earnings-hours schedule.

Using (15) and (10.1) the inverse of $\underline{\underline{V}}$ in (8) may be found explicitly. A by-product of the inversion is a more explicit form of (13); namely,

$$(18) \qquad \lambda_\alpha = \frac{\partial \lambda}{\partial \alpha} = \left[\text{determinant of } \underline{\underline{V}} \right]^{-1}$$

$$= \left\{ \underline{\underline{p}}^T \left[\underline{\underline{U}} - \frac{\underline{\underline{u}}_0 \, \underline{\underline{u}}_0^T}{u_{00} + \lambda\psi'} \right]^{-1} \underline{\underline{p}} \right.$$

$$\left. - \frac{2\psi \, \underline{\underline{p}}^T \underline{\underline{U}}^{-1} \underline{\underline{u}}_0}{u_{00} + \lambda\psi' - \underline{\underline{u}}_0^T \underline{\underline{U}}^{-1} \underline{\underline{u}}_0} + \frac{\psi^2}{u_{00} + \lambda\psi' - \underline{\underline{u}}_0^T \underline{\underline{U}}^{-1} \underline{\underline{u}}_0} \right\}^{-1} .$$

The inverse of $\underline{\underline{V}}$ itself is given in Table 1.

Table 1

Fundamental Inverse in the Theory of Labour Supply and Commodity Demand, $\underline{Y}^{-1}$

	first column	next n columns	last column
first row	$-\lambda_\alpha$ (see eqn (18)) 1x1	$\underline{x}_\alpha^T$ (see eqn (17.1)) 1xn	$y_{0\alpha}$ (see eqn (17.2))
next n rows	$\underline{x}_\alpha$ (see eqn(17.1)) nx1	$\left[\underline{I}_n - \underline{x}_\alpha \left(\underline{p}^T - \frac{\psi\, \underline{u}_0^T}{u_{00} + \lambda\psi'}\right)\right]\left[\underline{U} - \frac{\underline{u}_0 \underline{u}_0^T}{u_{00} + \lambda\psi'}\right]^{-1}$ nxn (see footnote 1)	$\frac{-1}{u_{00} + \lambda\psi' - \underline{u}_0^T \underline{U}^{-1} \underline{u}_0}\left[\underline{x}_\alpha (\psi - \underline{p}^T \underline{U}^{-1} \underline{u}_0) + \underline{U}^{-1} \underline{u}_0\right]$ nx1 (see footnote 1)
last row	$y_{0\alpha}$ (see eqn(17.2)) 1x1	$-\left[y_{0\alpha}\, \underline{p}^T + \frac{(1-y_{0\alpha}\psi)\, \underline{u}_0^T}{u_{00} + \lambda\psi'}\right]\left[\underline{U} - \frac{\underline{u}_0 \underline{u}_0^T}{u_{00} + \lambda\psi'}\right]^{-1}$ (see footnote 2)	$\frac{1}{u_{00} + \lambda\psi' - \underline{u}_0^T \underline{U}^{-1} \underline{u}_0}\left[(1-y_{0\alpha}\psi) + y_{0\alpha}\, \underline{p}^T \underline{U}^{-1} \underline{u}_0\right]$ 1x1 (see footnote 2)

1. $\underline{x}_\alpha$ may be substituted out of this expression using (17.1).
2. $y_{0\alpha}$ may be substituted out of this expression using (17.2).

3. RESPONSES TO COMMODITY PRICES

The solution of (8) is now examined term by term, starting with the first (super) column of $\underline{Y}$. The first term is the n-vector of derivatives of the marginal utility of money with respect to commodity prices:

$$(19) \qquad \underline{\lambda}_p^{\cdot} = -(\lambda_\alpha \underline{x}^T + \lambda \underline{x}_\alpha) .$$

Apart from the need to reinterpret λ_α along the lines of (18) - - in Barten, the corresponding formula is

$$(20) \qquad \lambda_\alpha = -\left[\underline{p}^T \underline{U}^{-1} \underline{p}\right]^{-1},$$

where α is total expenditure - - formula (19) is iso-morphic with the commodities only model [Barten (1964), p. 3]. If the only qualification needed for a result in this paper to be identical to the commodities only result is the reinterpretation of λ_α as above, then the result will be said to be "Barten isomorphic". Equation (19) is thus Barten-isomorphic. Continuing with (8)'s solution,

$$(21) \qquad \underline{X}_p = -\underline{x}_\alpha \underline{x}^T + \lambda \left[\underline{U} - \frac{\underline{u}_0 \underline{u}_0^T}{u_{00} + \lambda\psi'}\right]^{-1} - \frac{\lambda}{\lambda_\alpha} \underline{x}_\alpha \underline{x}_\alpha^T .$$

This is the generalization (of Barten's matrix generalization) of Hicks' (1939) fundamental equation of value theory. The first and third