1. Background

A payroll tax is levied only on larger firms -- those with wage bill > T. There are L of such firms and each pays a fraction M of the amount by which its wage bill exceeds T. The total wage bill of the L larger firms is B, so the total tax revenue R = M(B-LT). We call the term (B-LT), the tax base, or P: it plays a central role.

The problem is to find how tax revenue R is affected by a change in T. To begin with we assume that the number of firms and the wage bill of each firm is fixed (these assumptions will be relaxed later).

2. Notation and definitions

W	total wage bill in industry
M	marginal tax rate
T	threshold value
L	No of larger firms -- those paying tax
B	wage cost of firms with wage bill > T
R	tax revenue in industry = M.[B - LT]
P	Tax base = R/M = B - LT

If W B R M and T are known, the rest of the variables above could be deduced.

3. The Tax Base Function

Plainly, the tax base declines as the threshold value increases. We can define a function:

$P = \Pi(x)$ = tax base if threshold is set to x,

which we could sketch as follows.

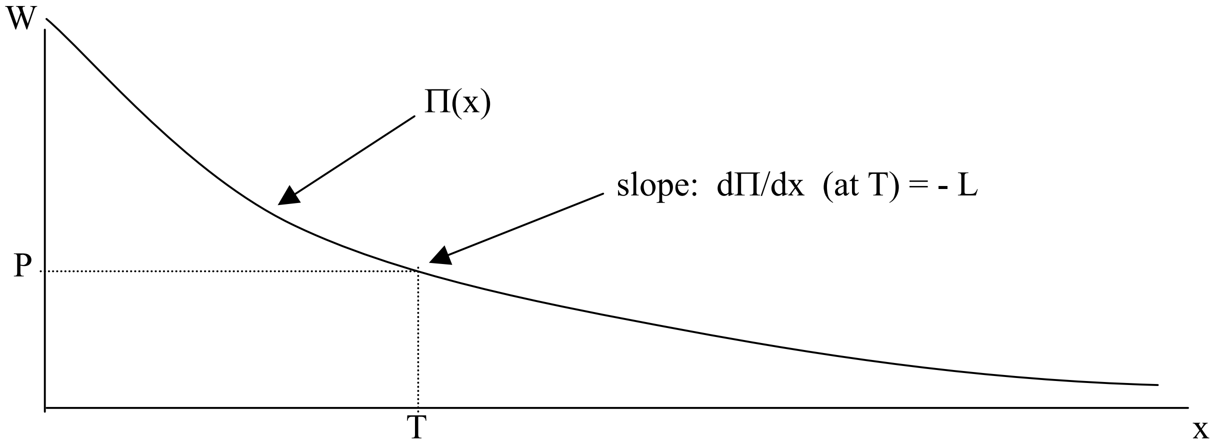

Diagram A: the tax base function

In Diagram A, note that:

$\Pi(0) = W$	since if the threshold x were zero, all wages would be taxable.
$\Pi(\infty) = 0$	since if the threshold x were high enough, no wages would be taxable.
$\Pi(T) = P$	observed in initial equilibrium

Now imagine the threshold was increased by a little bit dx. This has two effects:

(a) a few marginal firms fall below the threshold, and

(b) the L large firms each pay M.dx less tax, and P falls by L.dx.

However, the effect of (a) on either R or P may be ignored since marginal firms pay no tax anyway. This argument is sufficient to give the slope of Π at T:

$d\Pi/dx$ (at T) = - L more formally proved in Appendix A

Since L must be a declining function of T, Π must be convex to the origin, as drawn.

4. Initial Johansen Percent Change Implementation

From above we have:

$dP = - L.dT$

or $P.p = - L.T.t$, adopting % change notation

or $P.p = - (B-P).t$,

or $p = -\alpha t$ where $\alpha = [B-P]/P$ which can be calculated from observed shares.

Tax revenue is simply given by:

$r = m + p$

Interestingly, we can work out these first-order effects without knowing more about the shape of the Π function. For large change, however, we need a way to update B:

$B = LT + \Pi$

but we have seen that $L = -d\Pi/dx = - \Pi'$

so $B = \Pi - \Pi'.T$

so $B' = \Pi' - \Pi''.T - \Pi' = - \Pi''.T$

or $dB = - \Pi''.T.dT$

or $B.b = - \Pi''.T.T.t$ adopting % change notation

or $b = -\beta.t$ where $\beta = \Pi''.T.T/B$

To evaluate Π'' we need to hypothesize a functional form for Π that is consistent with the initial observed facts.

5. Fitting a functional form to the Tax Base function

We assume Π has the cumulative Weibull form:

$$\Pi(x) = W.\exp[-(x/a)^c]$$

Note that $\Pi(0)=W$ and $\Pi(\infty)=0$, as required. a and c are parameters which could be inferred from observed values of B and P. Our direct need is to evaluate $\beta = \Pi''.T.T/B$.

We differentiate twice:

$$\Pi' = d\Pi/dx = -\Pi(x).[c/a](x/a)^{c-1} = -\Pi(x).[c/x](x/a)^c$$

so $$\Pi'' = -\Pi'(x).[c/x](x/a)^c - \Pi(x).D\{[c/a](x/a)^{c-1}\}$$

$$= -\Pi'(x).[c/x](x/a)^c - \Pi(x).[c/a][(c-1)/a](x/a)^{c-2}$$

$= -\Pi'(x).[c/x](x/a)^c \quad - \quad \Pi(x).[c(c-1)/x^2](x/a)^c$

At T $\quad \Pi'' = -\Pi'.[c/T](T/a)^c \quad - \quad \Pi.[c(c-1)/T^2](T/a)^c$

We can make use of observed values of W, B, and P:

$P = \Pi(T) = W.\exp[-(T/a)^c] \qquad \text{so} \qquad \log(P/W) = -(T/a)^c$

$\Pi'(T) = -L = -\Pi(T).[c/T](T/a)^c = P.[c/T].\log(P/W)$

so $\quad LT = B - P = P.c.\log(W/P)$

So $\quad \Pi''(T) = L.[c/T]\log(W/P) \quad - \quad P.c.\log(W/P)(c-1)/T^2$

Use $\quad LT = P.c.\log(W/P)$

$\Pi''(T) = L.L.T[1/T]/P \quad - \quad LT(c-1)/T^2$

$= L.L/P \quad - \quad L(c-1)/T$

We want $\beta \;=\; \Pi''.T.T/B$

$= T.T.L.L/[P.B] \quad - \quad T.T.L(c-1)/[T.B]$

Define symbol $A = LT \qquad = B - P$ or total tax-free allowance

$\beta = A.A/[P.B] \quad - \quad A(c-1)/B$

$= A.A/[P.B] \quad + \quad A/B \quad - \quad A.c/B$

$= [A/B].\{1 + [A/P] - c\}$

Recall that $\quad LT = P.c.\log(W/P) \quad$ so $\quad c = [A/P]/\log(W/P)$

giving $\quad \beta = [A/B].\{1 + [A/P] - [A/P]/\log(W/P)\}$

$= [A/B].\{1 + [A/P][1 - 1/\log(W/P)]\} \qquad$ where $A = B - P$

6. Summary of Progress

We have deduced:

$p = -\alpha t$

$r = m + p$

$b = -\beta.t$

where α and β can each be calculated from observed shares as follows:

$\alpha = [B-P]/P = A/P \qquad$ where $A = B - P$

$\beta = [A/B].\{1 + \alpha[1 - 1/\log(W/P)]\}$

We assumed that all firms' wage bills were fixed.

7. Allowing for changes in firm size

We now allow for changes in the overall wage bill W. We assume still that the total number of firms and their *relative* sizes are fixed, so that if W increases by 1% each individual firm wage bill increases by 1%.

Our equations become:

$p = w-\alpha[t-w] \qquad$ *was* $\quad -\alpha t$

$r = m + p$ *unchanged*

$b = w-\beta[t-w]$ *was* $-\beta t$

Thus, if both the threshold T and W increase by 10%, the wage bill of the larger firms, B, and the tax base, P, also increase by 10%.

We can use similar reasoning to find the effect of a change in the total number of firms, N. We still assume that their relative sizes are fixed. Suppose each firm split into 2 halves (ie, N doubled) and the threshold was reduced by half -- B and P would be unchanged. Alternatively if both W and N increased by 1%, with T unchanged, B and P would increase by 1%. So:

$p = w-\alpha[n+t-w]$

$r = m + p$

$b = w-\beta[n+t-w]$

Obvious ways to model firm numbers would be to assume that N was fixed or that N followed real industry output:

$n = fn + z.$

T, which is a nominal, might either be exogenous, or follow a CPI-linked rule:

$t = ft + cpi$

8. GEMPACK Implementation

Add equations for each industry:

$p = w-\alpha[n+t-w]$

$r = m + p$

$b = w-\beta[n+t-w]$

$n = fn + z.$

$t = ft + cpi$

and define new variables p, r, b, n, t, fn and ft.

On file store new data:

B	wage cost of firms with wage bill > T	updated by	b
R	tax revenue in industry = M.[B - LT]	updated by	r
P	Tax base = R/M = B - LT	updated by	p

Define new coefficients α and β:

$\alpha = [B-P]/P$

$\beta = [(B-P)/B].\{1 + \alpha[1 - 1/\log(W/P)]\}$

Appendix A   Relating the Tax Base function to firm size distribution

In this section we relate the Tax Base function, Π, to some other concepts. The intention is to provide both some background for the results above, and a starting point for attacking similar problems.

We start by defining a density function f for the number of firms of a certain size. Let the number of firms with wage bill between A and A+dA be f(A).dA; equivalently, let the number of firms with wage bill between X and Y (Y>X) be:

$$\int_{A=X}^{Y} f(A).dA$$

It is convenient to define two additional functions:

let $Z(x) = \int_{A=x}^{\infty} f(A).dA$ = no of firms with wage bill > x = L

and $Q(x) = \int_{A=x}^{\infty} A.f(A).dA$ = wage bill of firms with wage bill > x = B

At the initial equilibrium we have:

$$dZ/dx = dL/dT = -f(T)$$

$$dQ/dx = dB/dT = -T.f(T) \qquad \text{(used to update B)}$$

then[1] $Q(x) = x.Z(x) + \int_{y=x}^{\infty} Z(y).dy$

So $\Pi(x) = \int_{y=x}^{\infty} Z(y).dy = Q(x) - x.Z(x)$ = tax base

We can define Z, Q and f in terms of Π as follows:

$$Z(x) = -d\Pi/dx \qquad \text{(giving change in P)}$$

$$Q(x) = x.Z(x) + \Pi(x)$$

$$f(x) = -dZ/dx = d^2\Pi/dx^2$$

In other words, Π is equivalent to an assumption about firm size distribution. Because we can derive Z, Q and f by differentiating Π, it turns out to be convenient to specify a form for Π. Had we specified a form for f or Z directly, some difficult integration might have been needed to find Π.

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[1] Proof that $Q(x) = x.Z(x) + \int_{y=x}^{\infty} Z(y).dy$ [a case of definite integration by parts]

d(LHS)/dx = -x.f(x) = d(RHS)/dx = -x.f(x) + Z(x) - Z(x)   So LHS -RHS = a constant K

let X = ∞ $Q(\infty) = \infty.Z(\infty) + \int_{y=\infty}^{\infty} Z(y).dy + K$

$0 = 0 + 0 + K$ so K= 0

Note that as x grows huge, x.Z(x) must tend to 0, if wage bill is to be finite.

Appendix B Sample computation results

Below are shown the results of adjusting T by either plus or minus 1% 325 times in succession, using the formulae of Section 6. The column headed "term" shows $\alpha[1 - 1/\log(W/P)]\}$.

Increasing threshold:

step	B	P	L	T	term	alpha	beta
1	80.00	50.00	30.00	1.000	-0.266	0.600	0.275
3	79.56	49.40	29.56	1.020	-0.255	0.610	0.282
6	78.88	48.49	28.91	1.051	-0.239	0.627	0.293
10	77.94	47.27	28.04	1.094	-0.217	0.649	0.308
15	76.72	45.73	26.95	1.149	-0.188	0.677	0.328
21	75.17	43.87	25.66	1.220	-0.152	0.714	0.353
28	73.27	41.66	24.16	1.308	-0.108	0.758	0.385
36	70.94	39.13	22.46	1.417	-0.053	0.813	0.424
45	68.15	36.26	20.58	1.549	0.013	0.879	0.474
55	64.81	33.08	18.54	1.711	0.092	0.959	0.535
66	60.87	29.61	16.37	1.909	0.188	1.056	0.610
78	56.27	25.90	14.12	2.152	0.305	1.173	0.704
91	50.98	22.03	11.82	2.449	0.445	1.314	0.821
105	45.01	18.11	9.56	2.815	0.616	1.486	0.966
120	38.42	14.26	7.40	3.268	0.825	1.695	1.148
136	31.41	10.64	5.42	3.832	1.081	1.953	1.376
153	24.26	7.42	3.71	4.538	1.398	2.271	1.665
171	17.39	4.74	2.33	5.428	1.791	2.666	2.030
190	11.30	2.72	1.31	6.558	2.285	3.162	2.495
210	6.45	1.35	0.64	8.001	2.910	3.789	3.093
231	3.08	0.55	0.26	9.861	3.709	4.592	3.867
253	1.16	0.18	0.08	12.274	4.744	5.632	4.878
276	0.32	0.04	0.02	15.430	6.106	7.000	6.218
300	0.06	0.01	0.00	19.593	7.932	8.834	8.024
325	0.00	0.00	0.00	25.126	10.438	11.352	10.512

Decreasing threshold:

step	B	P	L	T	term	alpha	beta
1	80.00	50.00	30.00	1.000	-0.266	0.600	0.275
3	80.44	50.60	30.45	0.980	-0.276	0.590	0.269
6	81.08	51.49	31.11	0.951	-0.291	0.575	0.259
10	81.91	52.67	32.00	0.914	-0.311	0.555	0.246
15	82.89	54.12	33.12	0.869	-0.334	0.532	0.231
21	84.01	55.83	34.45	0.818	-0.361	0.505	0.214
28	85.24	57.79	36.01	0.762	-0.391	0.475	0.196
36	86.52	59.95	37.77	0.703	-0.423	0.443	0.177
45	87.84	62.30	39.75	0.643	-0.456	0.410	0.158
55	89.17	64.80	41.92	0.581	-0.491	0.376	0.139
66	90.46	67.42	44.28	0.520	-0.525	0.342	0.121
78	91.69	70.10	46.80	0.461	-0.559	0.308	0.104
91	92.84	72.81	49.49	0.405	-0.592	0.275	0.088
105	93.90	75.51	52.31	0.352	-0.624	0.244	0.074
120	94.86	78.15	55.26	0.302	-0.653	0.214	0.061
136	95.71	80.70	58.31	0.257	-0.681	0.186	0.050
153	96.45	83.11	61.45	0.217	-0.707	0.160	0.041
171	97.09	85.37	64.67	0.181	-0.730	0.137	0.033
190	97.63	87.46	67.97	0.150	-0.751	0.116	0.026
210	98.08	89.35	71.33	0.122	-0.770	0.098	0.020
231	98.46	91.05	74.77	0.099	-0.787	0.081	0.016
253	98.77	92.55	78.27	0.079	-0.801	0.067	0.013
276	99.02	93.86	81.86	0.063	-0.813	0.055	0.010
300	99.23	94.99	85.55	0.050	-0.823	0.045	0.008
325	99.39	95.95	89.35	0.039	-0.832	0.036	0.006