Matrix Inverse

A square matrix $S \in \mathbb{R}^{n\times n}$ is invertible if there exists a matrix $S^{-1} \in \mathbb{R}^{n\times n}$ such that

$$S^{-1}S = I \quad \text{and} \quad SS^{-1} = I.$$

The matrix S^{-1} is called the inverse of S.

- An invertible matrix is also called non-singular.
  A matrix is called non-invertible or singular if it is not invertible.
- A matrix $S \in \mathbb{R}^{n\times n}$ cannot have two different inverses.
  In fact, if $X, Y \in \mathbb{R}^{n\times n}$ are two matrices with $XS = I$ and $SY = I$, then
  $$X = XI = X(SY) = (XS)Y = IY = Y.$$
- If $S \in \mathbb{R}^{n\times n}$ is invertible, then $Sx = f$ implies $x = S^{-1}Sx = S^{-1}f$, i.e., for every f the linear system $Sx = f$ has a solution $x = S^{-1}f$.
  The linear system $Sx = f$ cannot have more than one solution because $Sx = f$ and $Sy = f$ imply $S(x-y) = Sx - Sy = f - f = 0$ and $x - y = S^{-1}0 = 0$.
  Hence if S is invertible, then for every f the linear system $Sx = f$ has the unique solution $x = S^{-1}f$.
- We will see later that if for every f the linear system $Sx = f$ has a unique solution x, then S is invertible.

Computation of the Matrix Inverse

We want to find the inverse of $S \in \mathbb{R}^{n\times n}$, that is we want to find a matrix $X \in \mathbb{R}^{n\times n}$ such that $SX = I$.

- Let $X_{:,j}$ denote the jth column of X, i.e., $X = (X_{:,1}, \ldots, X_{:,n})$.
  Consider the matrix-matrix product SX. The jth column of SX is the matrix-vector product $SX_{:,j}$, i.e., $SX = (SX_{:,1}, \ldots, SX_{:,n})$.
  The jth column of the identity I is the jth unit vector $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T$.
  Hence $SX = (SX_{:,1}, \ldots, SX_{:,n}) = (e_1, \ldots, e_n) = I$ implies that we can compute the columns $X_{:,1}, \ldots, X_{:,n}$ of the inverse of S by solving n systems of linear equations
  $$\begin{aligned} SX_{:,1} &= e_1, \\ &\vdots \\ SX_{:,n} &= e_n. \end{aligned}$$
  Note that if for every f the linear system $Sx = f$ has a unique solution x, then there exists a unique $X = (X_{:,1}, \ldots, X_{:,n})$ with $SX = I$.

Want inverse of
$$S = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}.$$

Use Gaussian Elimination to solve the systems $SX_{:,1} = e_1, SX_{:,2} = e_2, SX_{:,3} = e_3$ for the three columns of $X = S^{-1}$

$$\left(\begin{array}{rrr|rrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|rrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right).$$

$$S^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}.$$

LU-Decomposition

Consider
$$S = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}.$$

Express Gaussian Elimination using Matrix-Matrix-multiplications

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}}_{=E_1} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}}_{=S} = \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{=E_1 S}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{=E_2} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{=E_1 S} = \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=E_2 E_1 S = U}$$

Inverses of E_1 and E_2 can be easily computed:

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

We have

$$E_2 E_1 S = U$$

Hence

$$E_1 S = E_2^{-1} U, \quad \text{and} \quad S = E_1^{-1} E_2^{-1} U.$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}}_{=S} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}}_{=E_1^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}_{=E_2^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix}}_{=E_1^{-1}E_2^{-1}=L} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U}$$

Hence we have the LU-Decomposition of S,

$$S = LU,$$

where L is a lower triangular matrix and U is an upper triangular matrix.
In Matlab compute using `[L,U]=lu(S)`.

If we have computed the LU decomposition

$$S = LU,$$

then we can use it to solve

$$Sx = f.$$

We replace S by LU,

$$LUx = f,$$

and introduce $y = Ux$. This leads to the two linear systems

$$Ly = f \quad \text{and} \quad Ux = y.$$

Since L is lower triangular and U is upper triangular, these two systems can be easily solved.
Example:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}}_{=S} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix}}_{=L} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U}, \quad f = \begin{pmatrix} -4 \\ -6 \\ -15 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix}}_{=L} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -4 \\ -6 \\ -15 \end{pmatrix}}_{=f} \quad \Rightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}}_{=y} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

In Matlab the matrix inverse is computed using the LU decomposition.
Given S, we want to compute S^{-1}. Recall that the columns $X_{:,1}, \ldots, X_{:,n}$ of the inverse $S^{-1} = X$ are the solutions of

$$SX_{:,1} = e_1,$$
$$\vdots$$
$$SX_{:,n} = e_n.$$

If we have computed the LU decomposition

$$S = LU,$$

then we can use it to solve the n linear systems $SX_{:,j} = e_j$, $j = 1, \ldots n$.
Use the LU decomposition to compute the inverse of

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}}_{=S} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix}}_{=L} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U}$$

What do you think of the following approach to solve $Sx = f$:

```
>>  Sinv = inv(S);
>>  x = Sinv*f;
```


Execute the following:

```
n    = input('Problem size = ');
S    = rand(n,n);
f    = rand(n,1);

ntry = 50;

tic
for i = 1:ntry
    x = S\f;
end
toc

tic
for i = 1:ntry
    Sinv = inv(S);
    x    = Sinv*f;
end
toc

tic
[L,U] =lu(S);
for i = 1:ntry
    x = U\(L\f);
end
toc
```